

STRICHARTZ ESTIMATES FOR THE FRACTIONAL SCHRÖDINGER AND WAVE EQUATIONS ON COMPACT MANIFOLDS WITHOUT BOUNDARY

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Abstract

We firstly prove Strichartz estimates for the fractional Schrödinger equations on \mathbb{R}^d , $d \geq 1$ endowed with a smooth bounded metric g . We then prove Strichartz estimates for the fractional Schrödinger and wave equations on compact Riemannian manifolds without boundary (M, g) . We finally give applications of Strichartz estimates for the local well-posedness of the pure power-type nonlinear fractional Schrödinger and wave equations posed on (M, g) .

Keywords: *Nonlinear fractional Schrödinger equation; Strichartz estimates; WKB approximation; pseudo-differential calculus; compact manifold.*

1 Introduction and main results

In the past several years, there has been much devotion to the understanding of fractional Schrödinger equation which is a fundamental equation of fractional quantum mechanics discovered by N. Laskin (see [24], [25]).

The Strichartz estimates play an important role in the study of nonlinear fractional Schrödinger equation on \mathbb{R}^d (see [17], [11], [19], [14] and references therein). Let us recall the local in time Strichartz estimates for the fractional Schrödinger operator on \mathbb{R}^d . For $\sigma \in (0, \infty) \setminus \{1\}$ and $I \subset \mathbb{R}$ a bounded interval, one has

$$\|e^{-it|D|^\sigma} u_0\|_{L^p(I, L^q(\mathbb{R}^d))} \leq C \|u_0\|_{H^{\gamma_{pq}}(\mathbb{R}^d)}, \quad (1.1)$$

where $|D| = \sqrt{-\Delta}$ with Δ is the free Laplace operator on \mathbb{R}^d and

$$\gamma_{pq} = \frac{d}{2} - \frac{d}{q} - \frac{\sigma}{p}$$

provided that (p, q) satisfies the Schrödinger admissible condition, namely

$$p \in [2, \infty], \quad q \in [2, \infty), \quad (p, q, d) \neq (2, \infty, 2), \quad \frac{2}{p} + \frac{d}{q} = \frac{d}{2}.$$

We refer to [14] for the general version of these Strichartz estimates on \mathbb{R}^d .

The main purpose of this paper is to prove Strichartz estimates for the fractional Schrödinger equation on \mathbb{R}^d equipped with a smooth bounded metric and on a compact manifold without boundary (M, g) .

Let us firstly consider \mathbb{R}^d endowed with a smooth Riemannian metric g . Let $g(x) = (g_{jk}(x))_{j,k=1}^d$ be a metric on \mathbb{R}^d , and denote $G(x) = (g^{jk}(x))_{j,k=1}^d := g^{-1}(x)$. The Laplace-Beltrami operator associated to g reads

$$\Delta_g = \sum_{j,k=1}^d |g(x)|^{-1} \partial_j (g^{jk}(x) |g(x)| \partial_k),$$

where $|g(x)| := \sqrt{\det g(x)}$ and denote $P := -\Delta_g$ the self-adjoint realization of $-\Delta_g$. Recall that the principal symbol of P is

$$p(x, \xi) = \xi^t G(x) \xi = \sum_{j,k=1}^d g^{jk}(x) \xi_j \xi_k.$$

In this paper, we assume that g satisfies the following assumptions.

1. There exists $C > 0$ such that for all $x, \xi \in \mathbb{R}^d$,

$$C^{-1} |\xi|^2 \leq \sum_{j,k=1}^d g^{jk}(x) \xi_j \xi_k \leq C |\xi|^2. \quad (1.2)$$

2. For all $\alpha \in \mathbb{N}^d$, there exists $C_\alpha > 0$ such that for all $x \in \mathbb{R}^d$,

$$|\partial^\alpha g^{jk}(x)| \leq C_\alpha, \quad j, k \in \{1, \dots, d\}. \quad (1.3)$$

We firstly note that the elliptic assumption (1.2) implies that $|g(x)|$ is bounded from below and above by positive constants. This shows that the space $L^q(\mathbb{R}^d, d\text{vol}_g)$, $1 \leq q \leq \infty$ where $d\text{vol}_g = |g(x)|dx$ and the usual Lebesgue space $L^q(\mathbb{R}^d)$ coincide. Thus in the sequel, the notation $L^q(\mathbb{R}^d)$ stands for either $L^q(\mathbb{R}^d, d\text{vol}_g)$ or the usual Lebesgue space $L^q(\mathbb{R}^d)$. It is well-known that under the assumptions (1.2) and (1.3), the Strichartz estimates (1.1) may fail at least for the Schrödinger equation (see [8], Appendix) and in this case (i.e. $\sigma = 2$) one has a loss of derivatives $1/p$ that is the right hand side of (1.1) is replaced by $\|u_0\|_{H^{1/p}(\mathbb{R}^d)}$. Here we extend the result of Burq-Gérard-Tzvetkov to the more general setting, i.e. $\sigma \in (0, \infty) \setminus \{1\}$ and obtain Strichartz estimates with a “loss” of derivatives $(\sigma - 1)/p$ when $\sigma \in (1, \infty)$ and without “loss” when $\sigma \in (0, 1)$. Throughout this paper, the “loss” compares to (1.1).

Theorem 1.1. *Consider \mathbb{R}^d , $d \geq 1$ equipped with a smooth metric g satisfying (1.2), (1.3) and let $I \subset \mathbb{R}$ a bounded interval. If $\sigma \in (1, \infty)$, then for all (p, q) Schrödinger admissible, there exists $C > 0$ such that for all $u_0 \in H^{\gamma_{pq} + (\sigma-1)/p}(\mathbb{R}^d)$,*

$$\|e^{-it|D_g|^\sigma} u_0\|_{L^p(I, L^q(\mathbb{R}^d))} \leq C \|u_0\|_{H^{\gamma_{pq} + (\sigma-1)/p}(\mathbb{R}^d)}, \quad (1.4)$$

where $|D_g| := \sqrt{P}$ and $\|u\|_{H^\gamma(\mathbb{R}^d)} := \|\langle D_g \rangle^\gamma u\|_{L^2(\mathbb{R}^d)}$. If $\sigma \in (0, 1)$, then (1.4) holds with $\gamma_{pq} + (\sigma - 1)/p$ is replaced by γ_{pq} .

The proof of (1.4) is based on the WKB approximation which is similar to [8]. Since we are working on manifolds, a good way is to decompose the semi-classical fractional Schrödinger operator, namely $e^{-ith^{-1}(h|D_g|)^\sigma}$, in the localized frequency, i.e. $e^{-ith^{-1}(h|D_g|)^\sigma} \varphi(h^2 P)$ for some $\varphi \in C_0^\infty(\mathbb{R} \setminus \{0\})$. The main difficulty is that in general we do not have the exact form of the semi-classical fractional Laplace-Beltrami operator in order to use the usual construction in [8]. To overcome this difficulty we write $e^{-ith^{-1}(h|D_g|)^\sigma} \varphi(h^2 P)$ as $e^{-ith^{-1}\psi(h^2 P)} \varphi(h^2 P)$ where $\psi(\lambda) = \tilde{\varphi}(\lambda) \sqrt{\lambda}^\sigma$ for some $\tilde{\varphi} \in C^\infty(\mathbb{R} \setminus \{0\})$ satisfying $\tilde{\varphi} = 1$ near $\text{supp}(\varphi)$. We then approximate $\psi(h^2 P)$ in terms of pseudo-differential operators and use the action of pseudo-differential operators on Fourier integral operators in order to construct an approximation for $e^{-ith^{-1}\psi(h^2 P)} \varphi(h^2 P)$. This approximation gives dispersive estimates for $e^{-ith^{-1}(h|D_g|)^\sigma} \varphi(h^2 P)$ on some small time interval independent of h . After scaling in time, we obtain Strichartz estimates without “loss” of derivatives over time intervals of size $h^{\sigma-1}$. When $\sigma \in (1, \infty)$, we can cumulate the bounded interval I by intervals of size $h^{\sigma-1}$ and get estimates with $(\sigma - 1)/p$ loss of derivatives. In the case $\sigma \in (0, 1)$, we can bound the estimates over time intervals of size 1 by the ones of size $h^{\sigma-1}$ and have the same Strichartz estimates as on \mathbb{R}^d . It is not a surprise that we recover the same Strichartz estimates as in the free case for $\sigma \in (0, 1)$ since $e^{it|D_g|^\sigma}$ has micro-locally the finite propagation speed property which is similar to $\sigma = 1$ for the (half) wave equation. Intuitively, if we consider the free Hamiltonian $H(x, \xi) = |\xi|^\sigma$, then the spatial component of geodesic

flow reads $x(t) = x(0) + t\sigma\xi|\xi|^{\sigma-2}$. After a time t , the distance $d(x(t), x(0)) \sim t|\xi|^{\sigma-1} \lesssim t$ if $\sigma - 1 \leq 0$ and $|\xi| \geq 1$. By decomposing the solution to $i\partial_t u - |D|^\sigma u = 0$ as $u = \sum_{k \geq 0} u_k$ where $u_k = \varphi(2^{-k}D)u$ is localized near $|\xi| \sim 2^k \geq 1$, we see that after a time t , all components u_k have traveled at a distance t from the data $u_k(0)$.

When \mathbb{R}^d is replaced by a compact Riemannian manifold without boundary (M, g) , Burq-Gérard-Tzvetkov established in [8] a Strichartz estimate with loss of $1/p$ derivatives for the Schrödinger equation, namely

$$\|e^{it\Delta_g} u_0\|_{L^p(I, L^q(M))} \leq C \|u_0\|_{H^{1/p}(M)}. \quad (1.5)$$

When M is the flat torus \mathbb{T}^d , Bourgain showed in [6], [7] some estimates related to (1.5) by means of the Fourier series for the Schrödinger equation. A direct consequence of these estimates is

$$\|e^{it\Delta_g} u_0\|_{L^4(\mathbb{T} \times \mathbb{T}^d)} \leq C \|u_0\|_{H^\gamma(\mathbb{T}^d)}, \quad \gamma > \frac{d}{4} - \frac{1}{2}.$$

Let us now consider the linear fractional Schrödinger equation posed on a compact Riemannian manifold without boundary (M, g) , namely

$$\begin{cases} i\partial_t u(t, m) - |D_g|^\sigma u(t, m) &= F(t, m), & (t, m) \in I \times M, \\ u(0, m) &= u_0(m), & m \in M, \end{cases} \quad (1.6)$$

where $|D_g| := \sqrt{-\Delta_g}$ with Δ_g is the Laplace-Beltrami operator on (M, g) . We have the following result.

Theorem 1.2. *Consider (M, g) a smooth compact boundaryless Riemannian manifold of dimension $d \geq 1$ and let $I \subset \mathbb{R}$ a bounded interval. If $\sigma \in (1, \infty)$, then for all (p, q) Schrödinger admissible, there exists $C > 0$ such that for all $u_0 \in H^{\gamma_{pq} + (\sigma-1)/p}(M)$,*

$$\|e^{-it|D_g|^\sigma} u_0\|_{L^p(I, L^q(M))} \leq C \|u_0\|_{H^{\gamma_{pq} + (\sigma-1)/p}(M)}. \quad (1.7)$$

Moreover, if u is a (weak) solution to (1.6), then

$$\|u\|_{L^p(I, L^q(M))} \leq C \left(\|u_0\|_{H^{\gamma_{pq} + (\sigma-1)/p}(M)} + \|F\|_{L^1(I, H^{\gamma_{pq} + (\sigma-1)/p}(M))} \right). \quad (1.8)$$

If $\sigma \in (0, 1)$, then (1.7) and (1.8) hold with γ_{pq} in place of $\gamma_{pq} + (\sigma - 1)/p$.

Remark 1.3. 1. Note that the exponents $\gamma_{pq} + (\sigma - 1)/p = d/2 - d/q - 1/p$ in the right hand side of (1.7) and $\gamma_{pq} = d/2 - d/q - \sigma/p$ in the case of $\sigma \in (0, 1)$ correspond to the gain of $1/p$ and σ/p derivatives respectively compared with the Sobolev embedding.

2. When $M = \mathbb{T}$ and $\sigma \in (1, 2)$, the authors in [13] established estimates related to (1.7), namely

$$\|e^{-it|D_g|^\sigma} u_0\|_{L^4(\mathbb{T} \times \mathbb{T})} \leq C \|u_0\|_{H^\gamma(\mathbb{T})}, \quad \gamma > \frac{2 - \sigma}{8}. \quad (1.9)$$

3. Using the same argument as in [8], we see that the endpoint homogeneous Strichartz estimate (1.7) are sharp on $\mathbb{S}^d, d \geq 3$. Indeed, let u_0 be a zonal spherical harmonic associated to eigenvalue $\lambda = k(d + k - 1)$. One has (see e.g. [29]) that for $\lambda \gg 1$,

$$\|u_0\|_{L^q(\mathbb{S}^d)} \sim \sqrt{\lambda}^{s(q)}, \quad s(q) = \frac{d-1}{2} - \frac{d}{q} \quad \text{if } \frac{2(d+1)}{d-1} \leq q \leq \infty.$$

Moreover, the above estimates are sharp. Therefore,

$$\|e^{-it|D_g|^\sigma} u_0\|_{L^2(I, L^{2^*}(\mathbb{S}^d))} = \|e^{-it\sqrt{\lambda}^\sigma} u_0\|_{L^2(I, L^{2^*}(\mathbb{S}^d))} \sim \sqrt{\lambda}^{s(2^*)},$$

where $2^* = 2d/(d-2)$ and $s(2^*) = 1/2$. This gives the optimality of (1.7) since $\gamma_{22^*} + (\sigma - 1)/2 = 1/2$.

A first application of Theorem 1.2 is the Strichartz estimates for the fractional wave equation posed on (M, g) . Let us consider the following linear fractional wave equation posed on (M, g) ,

$$\begin{cases} \partial_t^2 v(t, m) + |D_g|^{2\sigma} v(t, m) &= G(t, m), & (t, m) \in I \times M, \\ v(0, m) = v_0(m), & \partial_t v(0, m) = v_1(m), & m \in M. \end{cases} \quad (1.10)$$

We refer to [10] or [18] for the fractional wave equations.

Corollary 1.4. *Consider (M, g) a smooth compact boundaryless Riemannian manifold of dimension $d \geq 1$. Let $I \subset \mathbb{R}$ be a bounded interval and v a (weak) solution to (1.10). If $\sigma \in (1, \infty)$, then for all (p, q) Schrödinger admissible, there exists $C > 0$ such that for all $(v_0, v_1) \in H^{\gamma_{pq}+(\sigma-1)/p}(M) \times H^{\gamma_{pq}+(\sigma-1)/p-\sigma}(M)$,*

$$\|v\|_{L^p(I, L^q(M))} \leq C \left(\| [v](0) \|_{H^{\gamma_{pq}+(\sigma-1)/p}(M)} + \|G\|_{L^1(I, H^{\gamma_{pq}+(\sigma-1)/p-\sigma}(M))} \right), \quad (1.11)$$

where

$$\| [v](0) \|_{H^{\gamma_{pq}+(\sigma-1)/p}(M)} := \|v_0\|_{H^{\gamma_{pq}+(\sigma-1)/p}(M)} + \|v_1\|_{H^{\gamma_{pq}+(\sigma-1)/p-\sigma}(M)}.$$

If $\sigma \in (0, 1)$, then (1.11) holds with $\gamma_{pq} + (\sigma - 1)/p$ is replaced by γ_{pq} .

We next give applications of the Strichartz estimates given in Theorem 1.2. Let us consider the following nonlinear fractional Schrödinger equation

$$\begin{cases} i\partial_t u(t, m) - |D_g|^\sigma u(t, m) &= -\mu(|u|^{\nu-1}u)(t, m), & (t, m) \in I \times M, \mu \in \{\pm 1\}, \\ u(0, m) &= u_0(m), & m \in M. \end{cases} \quad (\text{NLFS})$$

with the exponent $\nu > 1$. The number $\mu = 1$ (resp. $\mu = -1$) corresponds to the defocusing case (resp. focusing case). By a standard approximation (see e.g. [16]), the following quantities are conserved by the flow of the equations,

$$\begin{aligned} M(u) &= \int_M |u(t, m)|^2 d\text{vol}_g(m), \\ E(u) &= \int_M \frac{1}{2} \| |D_g|^{\sigma/2} u(t, m) \|^2 + \frac{\mu}{\nu+1} |u(t, m)|^{\nu+1} d\text{vol}_g(m). \end{aligned}$$

Theorem 1.2 gives the following local well-posedness result.

Theorem 1.5. *Consider (M, g) a smooth compact boundaryless Riemannian manifold of dimension $d \geq 1$. Let $\sigma \in (1, \infty)$, $\nu > 1$ and $\gamma \geq 0$ be such that*

$$\begin{cases} \gamma > 1/2 - 1/\max(\nu - 1, 4) & \text{when } d = 1, \\ \gamma > d/2 - 1/\max(\nu - 1, 2) & \text{when } d \geq 2, \end{cases} \quad (1.12)$$

and also, if ν is not an odd integer,

$$\lceil \gamma \rceil \leq \nu, \quad (1.13)$$

where $\lceil \gamma \rceil$ is the smallest positive integer greater than or equal to γ . Then for all $u_0 \in H^\gamma(M)$, there exist $T > 0$ and a unique solution to (NLFS) satisfying

$$u \in C([0, T], H^\gamma(M)) \cap L^p([0, T], L^\infty(M)),$$

for some $p > \max(\nu - 1, 4)$ when $d = 1$ and some $p > \max(\nu - 1, 2)$ when $d \geq 2$. Moreover, the time T depends only on the size of the initial data, i.e. only on $\|u_0\|_{H^\gamma(M)}$. In the case $\sigma \in (0, 1)$, the same result holds with (1.12) is replaced by

$$\begin{cases} \gamma > 1/2 - \sigma/\max(\nu - 1, 4) & \text{when } d = 1, \\ \gamma > d/2 - \sigma/\max(\nu - 1, 2) & \text{when } d \geq 2. \end{cases} \quad (1.14)$$

We note that when ν is an odd integer, we have $F(\cdot) = -\mu|\cdot|^{\nu-1} \in C^\infty(\mathbb{C}, \mathbb{C})$ and when ν is not an odd integer, condition (1.13) implies $f \in C^{[\gamma]}(\mathbb{C}, \mathbb{C})$. It allows us to use the fractional derivatives (see [22], [14]).

As a direct consequence of Theorem 1.5 and the conservation laws, we have the following global well-posedness result for the defocusing nonlinear fractional Schrödinger equation, i.e. $\mu = 1$ in (NLFS).

Corollary 1.6. *Consider (M, g) a smooth compact boundaryless Riemannian manifold of dimension $d \geq 1$. Let $\sigma \in (1/2, \infty) \setminus \{1\}$ when $d = 1$, $\sigma > d - 1$ when $d \geq 2$ and $\nu > 1$ be such that if ν is not an odd integer, $\lceil \sigma/2 \rceil \leq \nu$. Then for all $u_0 \in H^{\sigma/2}(M)$, there exists a unique global solution $u \in C(\mathbb{R}, H^{\sigma/2}(M)) \cap L_{\text{loc}}^p(\mathbb{R}, L^\infty(M))$ to the defocusing (NLFS) for some $p > \max(\nu - 1, 4)$ when $d = 1$ and some $p > \max(\nu - 1, 2)$ when $d \geq 2$.*

We finally give applications of Strichartz estimates given in Corollary 1.4 for the nonlinear fractional wave equation. Let us consider the following nonlinear fractional wave equation posed on (M, g) ,

$$\begin{cases} \partial_t^2 v(t, m) + |D_g|^{2\sigma} v(t, m) &= -\mu(|v|^{\nu-1} v)(t, m), & (t, m) \in I \times M, \mu \in \{\pm 1\}, \\ v(0, m) = v_0(m), & \partial_t v(0, m) = v_1(m), & m \in M. \end{cases} \quad (\text{NLFW})$$

with $\sigma \in (0, \infty) \setminus \{1\}$ and the exponent $\nu > 1$. In this case, the following energy is conserved under the flow of the equation, i.e.

$$E(v, \partial_t v) = \int_M \frac{1}{2} |\partial_t v(t, m)|^2 + \frac{1}{2} |D_g|^\sigma v(t, m)|^2 + \frac{\mu}{\nu + 1} |v(t, m)|^{\nu+1} d\text{vol}_g(m).$$

Using the Strichartz estimates given in Corollary 1.4, we have the following local well-posedness result.

Theorem 1.7. *Consider (M, g) a smooth compact boundaryless Riemannian manifold of dimension $d \geq 1$. Let $\sigma \in (1, \infty)$, $\nu > 1$ and $\gamma \geq 0$ be as in (1.12) and also, if ν is not an odd integer, (1.13). Then for all $v_0 \in H^\gamma(M)$ and $v_1 \in H^{\gamma-\sigma}(M)$, there exist $T > 0$ and a unique solution to (NLFW) satisfying*

$$v \in C([0, T], H^\gamma(M)) \cap C^1([0, T], H^{\gamma-\sigma}(M)) \cap L^p([0, T], L^\infty(M)),$$

for some $p > \max(\nu - 1, 4)$ when $d = 1$ and some $p > \max(\nu - 1, 2)$ when $d \geq 2$. Moreover, the time T depends only on the size of the initial data, i.e. only on $\|v\|(0)_{H^\gamma(M)}$. In the case $\sigma \in (0, 1)$, the same result holds with (1.14) in place of (1.12).

We organize this paper as follows. In Section 2, we prove the Strichartz estimates on \mathbb{R}^d endowed with the smooth bounded metric g . In Section 3, we will give the proof of Strichartz estimates on compact manifolds (M, g) . We then prove the well-posedness results for the pure power-type of nonlinear fractional Schrödinger and wave equations on compact manifolds without boundary in Section 4.

Notation. In this paper the constant may change from line to line and will be denoted by the same C . The notation $A \lesssim B$ means that there exists $C > 0$ such that $A \leq CB$, and the one $A \sim B$ means that $A \lesssim B$ and $B \lesssim A$. For Banach spaces X and Y , the notation $\|\cdot\|_{\mathcal{L}(X, Y)}$ denotes the operator norm from X to Y and $\|\cdot\|_{\mathcal{L}(X)} := \|\cdot\|_{\mathcal{L}(X, X)}$.

2 Strichartz estimates on (\mathbb{R}^d, g)

2.1 Reduction of problem

In this subsection, we give a reduction of Theorem 1.1 due to the Littlewood-Paley decomposition. To do so, we firstly recall some useful facts on pseudo-differential calculus. For $m \in \mathbb{R}$, we consider the symbol class $S(m)$ the space of smooth functions a on \mathbb{R}^{2d} satisfying

$$|\partial_x^\alpha \partial_\xi^\beta a(x, \xi)| \leq C_{\alpha\beta} \langle \xi \rangle^{m-|\beta|},$$

for all $x, \xi \in \mathbb{R}^d$. We also need $S(-\infty) := \cap_{m \in \mathbb{R}} S(m)$. We define the semi-classical pseudo-differential operator with a symbol $a \in S(m)$ by

$$Op_h(a)u(x) := (2\pi h)^{-d} \iint_{\mathbb{R}^{2d}} e^{ih^{-1}(x-y)\xi} a(x, \xi) u(y) dy d\xi,$$

where $u \in \mathcal{S}(\mathbb{R}^d)$. The following result gives the $\mathcal{L}(L^q(\mathbb{R}^d), L^r(\mathbb{R}^d))$ -bound for pseudo-differential operators (see e.g. [5], Proposition 2.4).

Proposition 2.1. *Let $m > d$ and a be a continuous function on \mathbb{R}^{2d} smooth with respect to the second variable satisfying for all $\beta \in \mathbb{N}^d$, there exists $C_\beta > 0$ such that for all $x, \xi \in \mathbb{R}^d$,*

$$|\partial_\xi^\beta a(x, \xi)| \leq C_\beta \langle \xi \rangle^{-m}.$$

Then for all $1 \leq q \leq r \leq \infty$, there exists $C > 0$ such that for all $h \in (0, 1]$,

$$\|Op_h(a)\|_{\mathcal{L}(L^q(\mathbb{R}^d), L^r(\mathbb{R}^d))} \leq Ch^{-(\frac{d}{q} - \frac{d}{r})}.$$

For a given $f \in C_0^\infty(\mathbb{R})$, we can approximate $f(h^2P)$ in term of pseudo-differential operators. We have the following result (see e.g [5], [8] or [27]).

Proposition 2.2. *Consider \mathbb{R}^d equipped with a smooth metric g satisfying (1.2) and (1.3). Then for a given $f \in C_0^\infty(\mathbb{R})$, there exist a sequence of symbols $q_j \in S(-\infty)$ satisfying $q_0 = f \circ p$ and $\text{supp}(q_j) \subset \text{supp}(f \circ p)$ such that for all $N \geq 1$,*

$$f(h^2P) = \sum_{j=0}^{N-1} h^j Op_h(q_j) + h^N R_N(h),$$

and for all $m \geq 0$ and all $1 \leq q \leq r \leq \infty$, there exists $C > 0$ such that for all $h \in (0, 1]$,

$$\begin{aligned} \|R_N(h)\|_{\mathcal{L}(L^q(\mathbb{R}^d), L^r(\mathbb{R}^d))} &\leq Ch^{-(\frac{d}{q} - \frac{d}{r})}. \\ \|R_N(h)\|_{\mathcal{L}(H^{-m}(\mathbb{R}^d), H^m(\mathbb{R}^d))} &\leq Ch^{-2m}. \end{aligned}$$

A direct consequence of Proposition 2.1 and Proposition 2.2 is the following $\mathcal{L}(L^q(\mathbb{R}^d), L^r(\mathbb{R}^d))$ -bound for $f(h^2P)$.

Proposition 2.3. *Let $f \in C_0^\infty(\mathbb{R})$. Then for all $1 \leq q \leq r \leq \infty$, there exists $C > 0$ such that for all $h \in (0, 1]$,*

$$\|f(h^2P)\|_{\mathcal{L}(L^q(\mathbb{R}^d), L^r(\mathbb{R}^d))} \leq Ch^{-(\frac{d}{q} - \frac{d}{r})}.$$

Next, we need the following version of the Littlewood-Paley decomposition (see e.g. [8] or [5]).

Proposition 2.4. *There exist $\varphi_0 \in C_0^\infty(\mathbb{R})$ and $\varphi \in C_0^\infty(\mathbb{R} \setminus \{0\})$ such that*

$$\varphi_0(P) + \sum_{h^{-1}: \text{dya}} \varphi(h^2P) = \text{Id},$$

where $h^{-1} : \text{dya}$ means $h^{-1} = 2^k, k \in \mathbb{N} \setminus \{0\}$. Moreover, for all $q \in [2, \infty)$, there exists $C > 0$ such that for all $u \in \mathcal{S}(\mathbb{R}^d)$,

$$\|u\|_{L^q(\mathbb{R}^d)} \leq C \left(\sum_{h^{-1}: \text{dya}} \|\varphi(h^2P)u\|_{L^q(\mathbb{R}^d)}^2 \right)^{1/2} + C \|u\|_{L^2(\mathbb{R}^d)}.$$

We end this subsection with the following reduction.

Proposition 2.5. *Consider $\mathbb{R}^d, d \geq 1$ equipped with a smooth metric g satisfying (1.2), (1.3). Let $\sigma \in (0, \infty) \setminus \{1\}$ and $\varphi \in C_0^\infty(\mathbb{R} \setminus \{0\})$. If there exist $t_0 > 0$ small enough and $C > 0$ such that for all $u_0 \in L^1(\mathbb{R}^d)$ and all $h \in (0, 1]$,*

$$\|e^{-ith^{-1}(h|D_g|)^\sigma} \varphi(h^2 P) u_0\|_{L^\infty(\mathbb{R}^d)} \leq Ch^{-d/2} |t|^{-d/2} \|u_0\|_{L^1(\mathbb{R}^d)}, \quad (2.1)$$

for all $t \in [-t_0, t_0]$, then Theorem 1.1 holds true.

The proof of Proposition 2.5 bases on the following version of TT^* -criterion (see [23], [8] or [33]).

Theorem 2.6. *Let (X, \mathcal{M}, μ) be a σ -finite measured space, and $T : \mathbb{R} \rightarrow \mathcal{B}(L^2(X, \mathcal{M}, \mu))$ be a weakly measurable map satisfying, for some constants $C, \gamma, \delta > 0$,*

$$\|T(t)\|_{L^2(X) \rightarrow L^2(X)} \leq C, \quad t \in \mathbb{R}, \quad (2.2)$$

$$\|T(t)T(s)^*\|_{L^1(X) \rightarrow L^\infty(X)} \leq Ch^{-\gamma} |t-s|^{-\delta}, \quad t, s \in \mathbb{R}. \quad (2.3)$$

Then for all pair (p, q) satisfying

$$p \in [2, \infty], \quad q \in [1, \infty], \quad (p, q, \delta) \neq (2, \infty, 1), \quad \frac{2}{p} + \frac{2\delta}{q} = \delta,$$

one has

$$\|T(t)u\|_{L^p(\mathbb{R}, L^q(X))} \leq Ch^{-\kappa} \|u\|_{L^2(X)},$$

where $\kappa = (\gamma + \delta)(1/2 - 1/q) - 1/p$.

Proof of Proposition 2.5. Using the energy estimates and dispersive estimates (2.1), we can apply Theorem 2.6 for $T(t) = \mathbb{1}_{[-t_0, t_0]}(t) e^{-ith^{-1}(h|D_g|)^\sigma} \varphi(h^2 P)$, $\gamma = \delta = d/2$ and get

$$\|e^{-ith^{-1}(h|D_g|)^\sigma} \varphi(h^2 P) u_0\|_{L^p([-t_0, t_0], L^q(\mathbb{R}^d))} \leq Ch^{-(d/2 - d/q - 1/p)} \|u_0\|_{L^2(\mathbb{R}^d)}.$$

By scaling in time, we have

$$\begin{aligned} \|e^{-it|D_g|^\sigma} \varphi(h^2 P) u_0\|_{L^p(h^{\sigma-1}[-t_0, t_0], L^q(\mathbb{R}^d))} &= h^{(\sigma-1)/p} \|e^{-ith^{-1}(h|D_g|)^\sigma} \varphi(h^2 P) u_0\|_{L^p([-t_0, t_0], L^q(\mathbb{R}^d))} \\ &\leq Ch^{-\gamma_{pq}} \|u_0\|_{L^2(\mathbb{R}^d)}. \end{aligned} \quad (2.4)$$

Using the group property and the unitary property of Schrödinger operator $e^{it|D_g|^\sigma}$, we have the same estimates as in (2.4) for all intervals of size $2h^{\sigma-1}$. Indeed, for any interval I_h of size $2h^{\sigma-1}$, we can write $I_h = [c - h^{\sigma-1}t_0, c + h^{\sigma-1}t_0]$ for some $c \in \mathbb{R}$ and

$$\begin{aligned} \|e^{-it|D_g|^\sigma} \varphi(h^2 P) u_0\|_{L^p(I_h, L^q(\mathbb{R}^d))} &= \|e^{-it|D_g|^\sigma} \varphi(h^2 P) e^{-ic|D_g|^\sigma} u_0\|_{L^p(h^{\sigma-1}[-t_0, t_0], L^q(\mathbb{R}^d))} \\ &\leq Ch^{-\gamma_{pq}} \|e^{-ic|D_g|^\sigma} u_0\|_{L^2(\mathbb{R}^d)} = Ch^{-\gamma_{pq}} \|u_0\|_{L^2(\mathbb{R}^d)}. \end{aligned}$$

In the case $\sigma \in (1, \infty)$, we use a trick given in [8], i.e. cumulating $O(h^{1-\sigma})$ estimates on intervals of length $2h^{\sigma-1}$ to get estimates on any finite interval I . Precisely, by writing I as a union of N intervals I_h of length $2h^{\sigma-1}$ with $N \lesssim h^{1-\sigma}$, we have

$$\begin{aligned} \|e^{-it|D_g|^\sigma} \varphi(h^2 P) u_0\|_{L^p(I, L^q(\mathbb{R}^d))} &\leq \left(\sum_h \int_{I_h} \|e^{-it|D_g|^\sigma} \varphi(h^2 P) u_0\|_{L^q(\mathbb{R}^d)}^p dt \right)^{1/p} \\ &\leq CN^{1/p} h^{-\gamma_{pq}} \|u_0\|_{L^2(\mathbb{R}^d)} \leq Ch^{-\gamma_{pq} - (\sigma-1)/p} \|u_0\|_{L^2(\mathbb{R}^d)}. \end{aligned} \quad (2.5)$$

In the case $\sigma \in (0, 1)$, we can obviously bound the estimates over time intervals of size 1 by the ones of size $h^{\sigma-1}$ and obtain

$$\|e^{-it|D_g|^\sigma} \varphi(h^2 P) u_0\|_{L^p(I, L^q(\mathbb{R}^d))} \leq Ch^{-\gamma_{pq}} \|u_0\|_{L^2(\mathbb{R}^d)}. \quad (2.6)$$

Moreover, we can replace the norm $\|u_0\|_{L^2(\mathbb{R}^d)}$ in the right hand side of (2.5) and (2.6) by $\|\varphi(h^2P)u_0\|_{L^2(\mathbb{R}^d)}$. Indeed, by choosing $\tilde{\varphi} \in C_0^\infty(\mathbb{R} \setminus \{0\})$ satisfying $\tilde{\varphi} = 1$ near $\text{supp}(\varphi)$, we can write

$$e^{-ith^{-1}(h|D_g|)^\sigma} \varphi(h^2P)u_0 = e^{-ith^{-1}(h|D_g|)^\sigma} \tilde{\varphi}(h^2P)\varphi(h^2P)u_0$$

and apply (2.5) and (2.6) with $\tilde{\varphi}$ in place of φ . Now, by using the Littlewood-Paley decomposition given in Proposition 2.4 and the Minkowski inequality, we have for all (p, q) Schrödinger admissible,

$$\|u\|_{L^p(I, L^q(\mathbb{R}^d))} \leq C \left(\sum_{h^{-1}:\text{dya}} \|\varphi(h^2P)u\|_{L^p(I, L^q(\mathbb{R}^d))}^2 \right)^{1/2} + C\|u\|_{L^p(I, L^2(\mathbb{R}^d))}. \quad (2.7)$$

We now apply (2.7) for $u = e^{-it|D_g|^\sigma} u_0$ together with (2.5) and get for $\sigma \in (1, \infty)$,

$$\|e^{-it|D_g|^\sigma} u_0\|_{L^p(I, L^q(\mathbb{R}^d))} \leq C \left(\sum_{h^{-1}:\text{dya}} h^{-2(\gamma_{pq} + (\sigma-1)/p)} \|\varphi(h^2P)u_0\|_{L^2(\mathbb{R}^d)}^2 \right)^{1/2} + C\|u_0\|_{L^2(\mathbb{R}^d)}.$$

Here the boundedness of I is crucial to have a second bound in the right hand side. The almost orthogonality and the fact that $\gamma_{pq} + (\sigma-1)/p \geq 0$ imply for $\sigma \in (1, \infty)$,

$$\|e^{-it|D_g|^\sigma} u_0\|_{L^p(I, L^q(\mathbb{R}^d))} \leq C\|u_0\|_{H^{\gamma_{pq} + (\sigma-1)/p}(\mathbb{R}^d)}.$$

Similar results hold for $\sigma \in (0, 1)$ with γ_{pq} in place of $\gamma_{pq} + (\sigma-1)p$ by using (2.6) instead of (2.5). This completes the proof. \square

2.2 The WKB approximation

This subsection is devoted to the proof of dispersive estimates (2.1). To do so, we will use the so called WKB approximation (see [8], [5], [21] or [27]), i.e. to approximate $e^{-ith^{-1}(h|D_g|)^\sigma} \varphi(h^2P)$ in terms of Fourier integral operators. The following result is the main goal of this subsection. Let us denote $U_h(t) := e^{-ith^{-1}(h|D_g|)^\sigma}$ for simplifying the presentation.

Theorem 2.7. *Let $\sigma \in (0, \infty) \setminus \{1\}$, $\varphi \in C_0^\infty(\mathbb{R} \setminus \{0\})$, J a small neighborhood of $\text{supp}(\varphi)$ not containing the origin, $a \in S(-\infty)$ with $\text{supp}(a) \subset p^{-1}(\text{supp}(\varphi))$. Then there exist $t_0 > 0$ small enough, $S \in C^\infty([-t_0, t_0] \times \mathbb{R}^{2d})$ and a sequence of functions $a_j(t, \cdot, \cdot) \in S(-\infty)$ satisfying $\text{supp}(a_j(t, \cdot, \cdot)) \subset p^{-1}(J)$ uniformly with respect to $t \in [-t_0, t_0]$ such that for all $N \geq 1$,*

$$U_h(t)Op_h(a)u_0 = J_N(t)u_0 + R_N(t)u_0,$$

where

$$\begin{aligned} J_N(t)u_0(x) &= \sum_{j=0}^{N-1} h^j J_h(S(t), a_j(t))u_0(x) \\ &= \sum_{j=0}^{N-1} h^j \left[(2\pi h)^{-d} \iint_{\mathbb{R}^{2d}} e^{ih^{-1}(S(t, x, \xi) - y\xi)} a_j(t, x, \xi) u_0(y) dy d\xi \right], \end{aligned}$$

$J_N(0) = Op_h(a)$ and the remainder $R_N(t)$ satisfies for all $t \in [-t_0, t_0]$ and all $h \in (0, 1]$,

$$\|R_N(t)\|_{\mathcal{L}(L^2(\mathbb{R}^d))} \leq Ch^{N-1}. \quad (2.8)$$

Moreover, there exists a constant $C > 0$ such that for all $t \in [-t_0, t_0]$ and all $h \in (0, 1]$,

$$\|J_N(t)\|_{\mathcal{L}(L^1(\mathbb{R}^d), L^\infty(\mathbb{R}^d))} \leq Ch^{-d/2} |t|^{-d/2}. \quad (2.9)$$

Remark 2.8. Before entering to the proof of Theorem 2.7, let us prove (2.1). We firstly note that the study of dispersive estimates for $U_h(t)\varphi(h^2P)$ is reduced to the one of $U_h(t)Op_h(a)$ with $a \in S(-\infty)$ satisfying $\text{supp}(a) \subset p^{-1}(\text{supp}(\varphi))$. Indeed, by using the parametrix of $\varphi(h^2P)$ given in Proposition 2.2, we have for all $N \geq 1$,

$$\varphi(h^2P) = \sum_{j=0}^{N-1} h^j Op_h(\tilde{q}_j) + h^N \tilde{R}_N(h),$$

for some $\tilde{q}_j \in S(-\infty)$ satisfying $\text{supp}(\tilde{q}_j) \subset p^{-1}(\text{supp}(\varphi))$ and the remainder satisfies for all $m \geq 0$,

$$\|\tilde{R}_N(h)\|_{\mathcal{L}(H^{-m}(\mathbb{R}^d), H^m(\mathbb{R}^d))} \leq Ch^{-2m}.$$

Since $U_h(t)$ is bounded in $H^m(\mathbb{R}^d)$, the Sobolev embedding with $m > d/2$ implies

$$\|U_h(t)\tilde{R}_N(h)\|_{\mathcal{L}(L^1(\mathbb{R}^d), L^\infty(\mathbb{R}^d))} \leq \|U_h(t)\tilde{R}_N(h)\|_{\mathcal{L}(H^{-m}(\mathbb{R}^d), H^m(\mathbb{R}^d))} \leq Ch^{-2m}.$$

By choosing N large enough, the remainder term is bounded in $\mathcal{L}(L^1(\mathbb{R}^d), L^\infty(\mathbb{R}^d))$ independent of t, h . We next show that Theorem 2.7 gives dispersive estimates for $U_h(t)Op_h(a)$, i.e.

$$\|U_h(t)Op_h(a)\|_{\mathcal{L}(L^1(\mathbb{R}^d), L^\infty(\mathbb{R}^d))} \leq Ch^{-d/2}|t|^{-d/2}, \quad (2.10)$$

for all $h \in (0, 1]$ and all $t \in [-t_0, t_0]$. Indeed, by choosing $\tilde{\varphi} \in C_0^\infty(\mathbb{R} \setminus \{0\})$ which satisfies $\tilde{\varphi} = 1$ near $\text{supp}(\varphi)$, we can write

$$\begin{aligned} U_h(t)Op_h(a) &= \tilde{\varphi}(h^2P)U_h(t)Op_h(a)\tilde{\varphi}(h^2P) + (1 - \tilde{\varphi})(h^2P)U_h(t)Op_h(a)\tilde{\varphi}(h^2P) \\ &\quad + U_h(t)Op_h(a)(1 - \tilde{\varphi})(h^2P). \end{aligned} \quad (2.11)$$

Using Theorem 2.7, the first term is written as

$$\tilde{\varphi}(h^2P)U_h(t)Op_h(a)\tilde{\varphi}(h^2P) = \tilde{\varphi}(h^2P)J_N(t)\tilde{\varphi}(h^2P) + \tilde{\varphi}(h^2P)R_N(t)\tilde{\varphi}(h^2P).$$

We learn from Proposition 2.2 and (2.9) that the first term in the right hand side is of size $O_{\mathcal{L}(L^1(\mathbb{R}^d), L^\infty(\mathbb{R}^d))}(h^{-d/2}|t|^{-d/2})$ and the second one is of size $O_{\mathcal{L}(L^1(\mathbb{R}^d), L^\infty(\mathbb{R}^d))}(h^{N-1-d})$. For the second and the third term of (2.11), we compose to the left and the right hand side with $(P+1)^m$ for $m \geq 0$ and use the parametrix of $(1 - \tilde{\varphi})(h^2P)$. By composing pseudo-differential operators with disjoint supports, we obtain terms of size $O_{\mathcal{L}(L^2(\mathbb{R}^d))}(h^\infty)$. The Sobolev embedding with $m > d/2$ implies that the second and the third terms are of size $O_{\mathcal{L}(L^1(\mathbb{R}^d), L^\infty(\mathbb{R}^d))}(h^\infty)$. By choosing N large enough, we have (2.10).

Proof of Theorem 2.7. The proof is done by several steps.

Step 1: Construction of the phase. Due to the support of a , we can replace $(h|D_g|)^\sigma$ by $\psi(h^2P)$ where $\psi(\lambda) = \tilde{\varphi}\sqrt{\lambda}^\sigma$ with $\tilde{\varphi} \in C_0^\infty(\mathbb{R} \setminus \{0\})$ and $\tilde{\varphi} = 1$ on J . The interest of this replacement is that we can use Proposition 2.2 to write

$$\psi(h^2P) = \sum_{k=0}^{N-1} h^k Op_h(q_k) + h^N R_N(h), \quad (2.12)$$

where $q_k \in S(-\infty)$ satisfying $q_0(x, \xi) = \psi \circ p(x, \xi)$, $\text{supp}(q_k) \subset p^{-1}(\text{supp}(\psi))$ and $R_N(h)$ is bounded in $L^2(\mathbb{R}^d)$ uniformly in $h \in (0, 1]$. The standard Hamilton-Jacobi equation gives the following result (see e.g. [27] or Appendix A).

Proposition 2.9. *There exist $t_0 > 0$ small enough and a unique solution $S \in C^\infty([-t_0, t_0] \times \mathbb{R}^{2d})$ to the Hamilton-Jacobi equation*

$$\begin{cases} \partial_t S(t, x, \xi) + q_0(x, \nabla_x S(t, x, \xi)) &= 0, \\ S(0, x, \xi) &= x \cdot \xi. \end{cases} \quad (2.13)$$

Moreover, for all $\alpha, \beta \in \mathbb{N}^d$, there exists $C_{\alpha\beta} > 0$ such that for all $t \in [-t_0, t_0]$ and all $x, \xi \in \mathbb{R}^d$,

$$|\partial_x^\alpha \partial_\xi^\beta (S(t, x, \xi) - x \cdot \xi)| \leq C_{\alpha\beta}|t|, \quad |\alpha + \beta| \geq 1, \quad (2.14)$$

$$|\partial_x^\alpha \partial_\xi^\beta (S(t, x, \xi) - x \cdot \xi + tq_0(x, \xi))| \leq C_{\alpha\beta}|t|^2. \quad (2.15)$$

Step 2: Construction of amplitudes. The Duhamel formula yields

$$e^{-ith^{-1}\psi(h^2P)}Op_h(a)u_0 = J_N(t)u_0 - ih^{-1} \int_0^t e^{-i(t-s)h^{-1}\psi(h^2P)} (hD_s + \psi(h^2P)) J_N(s)u_0 ds.$$

We want the last term to have a small contribution. To do this, we need to consider the action of $hD_t + \psi(h^2P)$ on $J_N(t)$. We first compute the action of hD_t on $J_N(t)$ and have

$$hD_t \circ J_N(t) = \sum_{l=0}^N h^l J_h(S(t), b_l(t)),$$

where

$$\begin{aligned} b_0(t, x, \xi) &= \partial_t S(t, x, \xi) a_0(t, x, \xi), \\ b_l(t, x, \xi) &= \partial_t S(t, x, \xi) a_l(t, x, \xi) + D_t a_{l-1}(t, x, \xi), \quad l = 1, \dots, N-1, \\ b_N(t, x, \xi) &= D_t a_{N-1}(t, x, \xi). \end{aligned}$$

In order to study the action of $\psi(h^2P)$ on $J_N(t)$, we firstly need the parametrix of $\psi(h^2P)$ given in (2.12). We also need the following action of a pseudo-differential operator on a Fourier integral operator (see e.g. [27], [28] or [3], Appendix).

Proposition 2.10. *Let $b \in S(-\infty)$ and $c \in S(-\infty)$ and $S \in C^\infty(\mathbb{R}^{2d})$ satisfy for all $\alpha, \beta \in \mathbb{N}^d, |\alpha + \beta| \geq 1$, there exists $C_{\alpha\beta} > 0$,*

$$|\partial_x^\alpha \partial_\xi^\beta (S(x, \xi) - x \cdot \xi)| \leq C_{\alpha\beta}, \quad \forall x, \xi \in \mathbb{R}^d.$$

Then

$$Op_h(b) \circ J_h(S, c) = \sum_{j=0}^{N-1} h^j J_h(S, (b \triangleleft c)_j) + h^N J_h(S, r_N(h)),$$

where $(b \triangleleft c)_j$ is an universal linear combination of

$$\partial_\eta^\beta b(x, \partial_x S(x, \xi)) \partial_x^{\beta-\alpha} c(x, \xi) \partial_x^{\alpha_1} S(x, \xi) \cdots \partial_x^{\alpha_k} S(x, \xi),$$

with $\alpha \leq \beta, \alpha_1 + \cdots + \alpha_k = \alpha$ and $|\alpha_l| \geq 2$ for all $l = 1, \dots, k$ and $|\beta| = j$. The maps $(b, c) \mapsto (b \triangleleft c)_j$ and $(b, c) \mapsto r_N(h)$ are continuous from $S(-\infty) \times S(-\infty)$ to $S(-\infty)$ and $S(-\infty)$ respectively. In particular, we have

$$\begin{aligned} (b \triangleleft c)_0(x, \xi) &= b(x, \partial_x S(x, \xi)) c(x, \xi), \\ i(b \triangleleft c)_1(x, \xi) &= \partial_\eta b(x, \partial_x S(x, \xi)) \partial_x c(x, \xi) + \frac{1}{2} \text{tr} \left(\partial_{\eta, \eta}^2 b(x, \partial_x S(x, \xi)) \partial_{x, x}^2 S(x, \xi) \right) c(x, \xi). \end{aligned}$$

Using (2.12), Proposition 2.9, we can apply Proposition 2.10 and obtain

$$\begin{aligned} \psi(h^2P) \circ J_N(t) &= \sum_{k=0}^{N-1} h^k Op_h(q_k) \circ \sum_{j=0}^{N-1} h^j J_h(S(t), a_j(t)) + h^N R_N(h) J_N(t), \\ &= \sum_{k+j+l=0}^N h^{k+j+l} J_h(S(t), (q_k \triangleleft a_j(t))_l) + h^{N+1} J_h(S(t), r_{N+1}(h, t)) + h^N R_N(h) J_N(t), \end{aligned}$$

This implies that

$$(hD_t + \psi(h^2P)) J_N(t) = \sum_{r=0}^N h^r J_h(S(t), c_r(t)) + h^N R_N(h) J_N(t) + h^{N+1} J_h(S(t), r_{N+1}(h, t)),$$

where

$$\begin{aligned}
c_0(t) &= \partial_t S(t) a_0(t) + q_0(x, \nabla_x S(t)) a_0(t), \\
c_r(t) &= \partial_t S(t) a_r(t) + q_0(x, \nabla_x S(t)) a_r(t) + D_t a_{r-1}(t) + (q_0 \triangleleft a_{r-1}(t))_1 + (q_1 \triangleleft a_{r-1}(t))_0 \\
&\quad + \sum_{\substack{k+j+l=r \\ j \leq r-2}} (q_k \triangleleft a_j(t))_l, \quad r = 1, \dots, N-1, \\
c_N(t) &= D_t a_{N-1}(t) + (q_0 \triangleleft a_{N-1}(t))_1 + (q_1 \triangleleft a_{N-1}(t))_0 + \sum_{\substack{k+j+l=N \\ j \leq N-2}} (q_k \triangleleft a_j(t))_l.
\end{aligned}$$

Thanks to the Hamilton-Jacobi equation given in Proposition 2.9, the system of equations $c_r(t) = 0$ for $r = 0, \dots, N$ leads to the following transport equations

$$D_t a_0(t, x, \xi) + (q_0 \triangleleft a_0(t))_1 + (q_1 \triangleleft a_0(t))_0 = 0, \quad (2.16)$$

$$D_t a_r(t, x, \xi) + (q_0 \triangleleft a_r(t))_1 + (q_1 \triangleleft a_r(t))_0 = - \sum_{\substack{k+j+l=r+1 \\ j \leq r-1}} (q_k \triangleleft a_j(t))_l, \quad (2.17)$$

for $r = 1, \dots, N-1$ with initial data

$$a_0(0, x, \xi) = a(x, \xi), \quad a_r(0, x, \xi) = 0, \quad r = 1, \dots, N-1. \quad (2.18)$$

We can rewrite these equations as

$$\begin{aligned}
\partial_t a_0(t, x, \xi) + V(t, x, \xi) \cdot \nabla_x a_0(t, x, \xi) + f(t, x, \xi) a_0(t, x, \xi) &= 0, \\
\partial_t a_r(t, x, \xi) + V(t, x, \xi) \cdot \nabla_x a_r(t, x, \xi) + f(t, x, \xi) a_r(t, x, \xi) &= g_r(t, x, \xi),
\end{aligned}$$

for $r = 1, \dots, N-1$ where

$$\begin{aligned}
V(t, x, \xi) &= (\partial_\xi q_0)(x, \nabla_x S(t, x, \xi)), \\
f(t, x, \xi) &= \frac{1}{2} \text{tr} [(\partial_\xi^2 q_0)(x, \nabla_x S(t, x, \xi)) \cdot (\partial_x^2 S)(t, x, \xi)] + i q_1(x, \nabla_x S(t, x, \xi)), \\
g_r(t, x, \xi) &= -i \sum_{\substack{k+j+l=r+1 \\ j \leq r-1}} (q_k \triangleleft a_j(t))_l.
\end{aligned}$$

We now construct $a_r(t, x, \xi), r = 0, \dots, N-1$ by the method of characteristics as follows. Let $Z(t, s, x, \xi)$ be the flow associated to $V(t, x, \xi)$, i.e.

$$\partial_t Z(t, s, x, \xi) = V(t, Z(t, s, x, \xi), \xi), \quad Z(s, s, x, \xi) = x.$$

By the fact that $q_0 \in S(-\infty)$ and (2.14) and using the same trick as in Lemma A.1, we have

$$|\partial_x^\alpha \partial_\xi^\beta (Z(t, s, x, \xi) - x)| \leq C_{\alpha\beta} |t - s|, \quad (2.19)$$

for all $|t|, |s| \leq t_0$. Now, we can define iteratively

$$\begin{aligned}
a_0(t, x, \xi) &= a(Z(0, t, x, \xi), \xi) \exp \left(- \int_0^t f(s, Z(s, t, x, \xi), \xi) ds \right), \\
a_r(t, x, \xi) &= \int_0^t g_r(s, Z(s, t, x, \xi), \xi) \exp \left(- \int_\tau^t f(\tau, Z(\tau, t, x, \xi), \xi) d\tau \right) ds,
\end{aligned}$$

for $r = 1, \dots, N-1$. These functions are respectively solutions to (2.16) and (2.17) with initial data (2.18) respectively. Since $\text{supp}(a) \subset p^{-1}(\text{supp}(\varphi))$, we see that for $t_0 > 0$ small enough, $(Z(t, s, p^{-1}(\text{supp}(\varphi))), \xi) \in p^{-1}(J)$ for all $|t|, |s| \leq t_0$. By extending $a_r(t, x, \xi)$ on \mathbb{R}^{2d} by $a_r(t, x, \xi) = 0$ for $(x, \xi) \notin p^{-1}(J)$, the functions a_r are still smooth in $(x, \xi) \in \mathbb{R}^{2d}$. Using the fact that $a, q_k \in S(-\infty)$, (2.15) and (2.19), we have for $t_0 > 0$ small enough, $a_r(t, \cdot, \cdot)$ is a bounded set of $S(-\infty)$ and $\text{supp}(a_r(t, \cdot, \cdot)) \in p^{-1}(J)$ uniformly with respect to $t \in [-t_0, t_0]$.

Step 3: L^2 -boundedness of remainder. We will use the so called Kuranishi trick (see e.g. [27], [26]). We firstly have

$$R_N(t) = -ih^{N-1} \int_0^t e^{-i(t-s)h^{-1}\psi(h^2P)} \left(R_N(h)J_N(s) + hJ_h(S(s), r_{N+1}(h, s)) \right) ds.$$

Using that $e^{-i(t-s)h^{-1}\psi(h^2P)}$ is unitary in $L^2(\mathbb{R}^d)$ and Proposition 2.2 that $R_N(h)$ is bounded in $\mathcal{L}(L^2(\mathbb{R}^d))$ uniformly in $h \in (0, 1]$, the estimate (2.8) follows from the L^2 -boundedness of $J_h(S(t), a(t))$ uniformly with respect to $h \in (0, 1]$ and $t \in [-t_0, t_0]$ where $(a(t))_{t \in [-t_0, t_0]}$ is bounded in $S(-\infty)$. For $t \in [-t_0, t_0]$, we define a map on \mathbb{R}^{3d} by

$$\Lambda(t, x, y, \xi) := \int_0^1 \nabla_x S(t, y + s(x - y), \xi) ds.$$

Using (2.14), we have for $t_0 > 0$ small enough,

$$|\nabla_x \cdot \nabla_\xi S(t, x, \xi) - I_{\mathbb{R}^d}| \ll 1, \quad \forall x, \xi \in \mathbb{R}^d.$$

This implies that

$$|\nabla_\xi \Lambda(t, x, y, \xi)| \leq \int_0^1 |\nabla_\xi \cdot \nabla_x S(t, y + s(x - y), \xi)| ds \ll 1, \quad \forall t \in [-t_0, t_0].$$

Thus for all $t \in [-t_0, t_0]$ and all $x, y \in \mathbb{R}^d$, the map $\xi \mapsto \Lambda(t, x, y, \xi)$ is a diffeomorphism from \mathbb{R}^d onto itself. If we denote $\xi \mapsto \Lambda^{-1}(t, x, y, \xi)$ the inverse map, then $\Lambda^{-1}(t, x, y, \xi)$ satisfies (see [3]) that: for all $\alpha, \alpha', \beta \in \mathbb{N}^d$, there exists $C_{\alpha\alpha'\beta} > 0$ such that

$$|\partial_x^\alpha \partial_y^{\alpha'} \partial_\xi^\beta (\Lambda^{-1}(t, x, y, \xi) - \xi)| \leq C_{\alpha\alpha'\beta} |t|, \quad (2.20)$$

for all $t \in [-t_0, t_0]$. Now, by change of variable $\xi \mapsto \Lambda^{-1}(t, x, y, \xi)$, the action $J_h(S(t), a(t)) \circ J_h(S(t), a(t))^*$ becomes (see [27]) a semi-classical pseudo-differential operator with the amplitude

$$a(t, x, \Lambda^{-1}(t, x, y, \xi)) \overline{a(t, y, \Lambda^{-1}(t, x, y, \xi))} |\det \partial_\xi \Lambda^{-1}(t, x, y, \xi)|.$$

Using the fact that $(a(t))_{t \in [-t_0, t_0]}$ is bounded in $S(-\infty)$ and (2.20), this amplitude and its derivatives are bounded. By the Calderón-Vaillancourt theorem, we have the result.

Step 4: Dispersive estimates. We prove the result for a general term, namely $J_h(S(t), a(t))$ with $(a(t))_{t \in [-t_0, t_0]}$ is bounded in $S(-\infty)$ satisfying $\text{supp}(a(t, \cdot, \cdot)) \in p^{-1}(J)$ for some small neighborhood J of $\text{supp}(\varphi)$ not containing the origin uniformly with respect to $t \in [-t_0, t_0]$. The kernel of $J_h(S(t), a(t))$ reads

$$K_h(t, x, y) = (2\pi h)^{-d} \int_{\mathbb{R}^d} e^{ih^{-1}(S(t, x, \xi) - y\xi)} a(t, x, \xi) d\xi.$$

It suffices to show for all $t \in [-t_0, t_0]$ and all $h \in (0, 1]$, $|K_h(t, x, y)| \leq Ch^{-d/2} |t|^{-d/2}$, for all $x, y \in \mathbb{R}^d$. We only consider the case $t \geq 0$, for $t \leq 0$ it is similar. Since the amplitude is compactly supported in ξ and $a(t, x, \xi)$ is bounded uniformly in $t \in [-t_0, t_0]$ and $x, y \in \mathbb{R}^d$, we have $|K_h(t, x, y)| \leq Ch^{-d}$. If $0 \leq t \leq h$ or $th^{-1} \leq 1$, then

$$|K_h(t, x, y)| \leq Ch^{-d} \leq Ch^{-d}(th^{-1})^{-d/2} = Ch^{-d/2} t^{-d/2}.$$

We now can assume that $h \leq t \leq t_0$ and write the phase function as $(S(t, x, \xi) - y\xi)/t$ with the parameter $\lambda = th^{-1} \geq 1$. By the choice of $\tilde{\varphi}$ (see Step 1 for $\tilde{\varphi}$), we see that on the support of the amplitude, i.e. on $p^{-1}(J)$, $q_0(x, \xi) = \sqrt{p(x, \xi)}^\sigma$. Thus we apply (2.15) to write

$$S(t, x, \xi) = x \cdot \xi - t \sqrt{p(x, \xi)}^\sigma + t^2 \int_0^1 (1 - \theta) \partial_t^2 S(\theta t, x, \xi) d\theta.$$

Next, using that $p(x, \xi) = \xi^t G(x) \xi = |\eta|^2$ with $\eta = \sqrt{G(x)} \xi$ or $\xi = \sqrt{g(x)} \eta$ where $g(x) = (g_{jk}(x))_{j,k=1}^d$ and $G(x) = (g(x))^{-1} = (g^{jk}(x))_{j,k=1}^d$, the kernel can be written as

$$K_h(t, x, y) = (2\pi h)^{-d} \int_{\mathbb{R}^d} e^{i\lambda \Phi(t, x, y, \eta)} a(t, x, \sqrt{g(x)} \eta) |g(x)| d\eta,$$

where

$$\Phi(t, x, y, \eta) = \frac{\sqrt{g(x)}(x-y) \cdot \eta}{t} - |\eta|^\sigma + t \int_0^1 (1-\theta) \partial_t^2 S(\theta t, x, \sqrt{g(x)} \eta) d\theta.$$

Recall that $|g(x)| := \sqrt{\det g(x)}$. By (1.2), $|||\sqrt{G(x)}|||$ and $|||\sqrt{g(x)}|||$ are bounded from below and above uniformly in $x \in \mathbb{R}^d$. This implies that η still belongs to a compact set of \mathbb{R}^d away from zero. We denote this compact support by \mathcal{K} . The gradient of the phase is

$$\nabla_\eta \Phi(t, x, y, \eta) = \frac{\sqrt{g(x)}(x-y)}{t} - \sigma \eta |\eta|^{\sigma-2} + t \left(\int_0^1 (1-\theta) (\nabla_\xi \partial_t^2 S)(\theta t, x, \sqrt{g(x)} \eta) d\theta \right) \sqrt{g(x)}.$$

Let us consider the case $|\sqrt{g(x)}(x-y)/t| \geq C$ for some constant C large enough. Thanks to the Hamilton-Jacobi equation (2.13) (see also (A.9), (A.2) and Lemma A.2) and the fact $\sigma \in (0, \infty) \setminus \{1\}$, we have for t_0 small enough,

$$|\nabla_\eta \Phi| \geq |\sqrt{g(x)}(x-y)/t| - \sigma |\eta|^{\sigma-1} - O(t) \geq C_1.$$

Hence we can apply the non stationary theorem, i.e. by integrating by parts with respect to η together with the fact that for all $\beta \in \mathbb{N}^d$ satisfying $|\beta| \geq 2$, $|\partial_\eta^\beta \Phi(t, x, y, \eta)| \leq C_\beta$, we have for all $N \geq 1$,

$$|K_h(t, x, y)| \leq C h^{-d} \lambda^{-N} = C h^{-d/2} t^{-d/2},$$

provided N is taken greater than $d/2$.

Thus we can assume that $|\sqrt{g(x)}(x-y)/t| \leq C$. In this case, we write

$$\nabla_\eta^2 \Phi(t, x, y, \eta) = -\sigma |\eta|^{\sigma-2} \left(I_{\mathbb{R}^d} + (\sigma-2) \frac{\eta \cdot \eta^t}{|\eta|^2} \right) + O(t).$$

Using that

$$\left| \det \sigma |\eta|^{\sigma-2} \left(I_{\mathbb{R}^d} + (\sigma-2) \frac{\eta \cdot \eta^t}{|\eta|^2} \right) \right| = \sigma^d |\sigma-1| |\eta|^{(\sigma-2)d} \geq C.$$

Therefore, for $t_0 > 0$ small enough, the map $\eta \mapsto \nabla_\eta \Phi(t, x, y, \eta)$ from a neighborhood of \mathcal{K} to its range is a local diffeomorphism. Moreover, for all $\beta \in \mathbb{N}^d$ satisfying $|\beta| \geq 1$, we have $|\partial_\eta^\beta \Phi(t, x, y, \eta)| \leq C_\beta$. The stationary phase theorem then implies that for all $t \in [h, t_0]$ and all $x, y \in \mathbb{R}^d$ satisfying $|\sqrt{g(x)}(x-y)/t| \leq C$,

$$|K_h(t, x, y)| \leq C h^{-d} \lambda^{-d/2} \leq C h^{-d/2} t^{-d/2}.$$

This completes the proof. \square

3 Strichartz estimates on compact manifolds

In this section, we give the proof of Strichartz estimates on compact manifolds without boundary given in Theorem 1.2.

3.1 Notations

Coordinate charts and partition of unity. Let M be a smooth compact Riemannian manifold without boundary. A coordinate chart $(U_\kappa, V_\kappa, \kappa)$ on M comprises an homeomorphism κ between an open subset U_κ of M and an open subset V_κ of \mathbb{R}^d . Given $\phi \in C_0^\infty(U_\kappa)$ (resp. $\chi \in C_0^\infty(V_\kappa)$), we define the pushforward of ϕ (resp. pullback of χ) by $\kappa_* \phi := \phi \circ \kappa^{-1}$ (resp. $\kappa^* \chi := \chi \circ \kappa$). For a given finite cover of M , namely $M = \cup_{\kappa \in \mathcal{F}} U_\kappa$ with $\#\mathcal{F} < \infty$, there exist $\phi_\kappa \in C_0^\infty(U_\kappa)$, $\kappa \in \mathcal{F}$ such that $1 = \sum_{\kappa} \phi_\kappa(m)$ for all $m \in M$.

Laplace-Beltrami operator. For all coordinate chart $(U_\kappa, V_\kappa, \kappa)$, there exists a symmetric positive definite matrix $g_\kappa(x) := (g_{jk}^\kappa(x))_{j,k=1}^d$ with smooth and real valued coefficients on V_κ such that the Laplace-Beltrami operator $P = -\Delta_g$ reads in $(U_\kappa, V_\kappa, \kappa)$ as

$$P_\kappa := -\kappa_* \Delta_g \kappa^* = - \sum_{j,k=1}^d |g_\kappa(x)|^{-1} \partial_j \left(|g_\kappa(x)| g_\kappa^{jk}(x) \partial_k \right),$$

where $|g_\kappa(x)| = \sqrt{\det g_\kappa(x)}$ and $(g_\kappa^{jk}(x))_{j,k=1}^d := (g_\kappa(x))^{-1}$. The principal symbol of P_κ is

$$p_\kappa(x, \xi) = \sum_{j,k=1}^d g_\kappa^{jk}(x) \xi_j \xi_k.$$

3.2 Functional calculus

In this subsection, we recall well-known facts on pseudo-differential calculus on manifolds (see e.g. [8]). For a given $a \in S(m)$, we define the operator

$$Op_h^\kappa(a) := \kappa^* Op_h(a) \kappa_*. \quad (3.1)$$

If nothing is specified about $a \in S(m)$, then the operator $Op_h^\kappa(a)$ maps $C_0^\infty(U_\kappa)$ to $C^\infty(U_\kappa)$. In the case $\text{supp}(a) \subset V_\kappa \times \mathbb{R}^d$, we have that $Op_h^\kappa(a)$ maps $C_0^\infty(U_\kappa)$ to $C_0^\infty(U_\kappa)$ hence to $C^\infty(M)$. We have the following result.

Proposition 3.1. *Let $\phi_\kappa \in C_0^\infty(U_\kappa)$ be an element of a partition of unity on M and $\tilde{\phi}_\kappa, \tilde{\tilde{\phi}}_\kappa \in C_0^\infty(U_\kappa)$ be such that $\tilde{\phi}_\kappa = 1$ near $\text{supp}(\phi_\kappa)$ and $\tilde{\tilde{\phi}}_\kappa = 1$ near $\text{supp}(\tilde{\phi}_\kappa)$. Then for all $N \geq 1$, all $z \in [0, +\infty)$ and all $h \in (0, 1]$,*

$$(h^2 P - z)^{-1} \phi_\kappa = \sum_{j=0}^{N-1} h^j \tilde{\phi}_\kappa Op_h^\kappa(q_{\kappa,j}(z)) \phi_\kappa + h^N R_N(z, h),$$

where $q_{\kappa,j}(z) \in S(-2-j)$ is a linear combination of $a_k(p_\kappa - z)^{-1-k}$ for some symbol $a_k \in S(2k-j)$ independent of z and

$$R_N(z, h) = -(h^2 P - z)^{-1} \tilde{\tilde{\phi}}_\kappa Op_h^\kappa(r_{\kappa,N}(z, h)) \phi_\kappa,$$

where $r_{\kappa,N}(z, h) \in S(-N)$ with seminorms growing polynomially in $1/\text{dist}(z, \mathbb{R}^+)$ uniformly in $h \in (0, 1]$ as long as z belongs to a bounded set of $\mathbb{C} \setminus [0, +\infty)$.

Proof. Let us set $\chi_\kappa := \kappa_* \phi_\kappa$, similarly for $\tilde{\chi}_\kappa$ and $\tilde{\tilde{\chi}}_\kappa$ and get $\chi_\kappa, \tilde{\chi}_\kappa, \tilde{\tilde{\chi}}_\kappa \in C_0^\infty(V_\kappa)$ and $\tilde{\chi}_\kappa = 1$ near $\text{supp}(\chi_\kappa)$ and $\tilde{\tilde{\chi}}_\kappa = 1$ near $\text{supp}(\tilde{\chi}_\kappa)$. We firstly find an operator, still denoted by P , globally defined on \mathbb{R}^d of the form

$$P = - \sum_{j,k=1}^d g^{jk}(x) \partial_j \partial_k + \sum_{l=1}^d b_l(x) \partial_l, \quad (3.2)$$

which coincides with P_κ on a large relatively compact subset V_0 of V_κ . By “large”, we mean that $\text{supp}(\tilde{\tilde{\chi}}_\kappa) \subset V_0$. For instance, we can take $P = v P_\kappa - (1-v) \Delta$ where $v \in C_0^\infty(V_\kappa)$ with values in $[0, 1]$ satisfying $v = 1$ on V_0 . The principal symbol of P is

$$p(x, \xi) = \sum_{j,k=1}^d g^{jk}(x) \xi_j \xi_k, \quad \text{where } g^{jk}(x) = v(x) g_\kappa^{jk}(x) + (1-v(x)) \delta_{jk}. \quad (3.3)$$

It is easy to see that $g(x) = (g^{jk}(x))$ satisfies (1.2) and (1.3) and b_l is bounded in \mathbb{R}^d together with all of its derivatives. Using the standard elliptic parametrix for $(h^2 P - z)^{-1}$ (see e.g [4], [27]), we have

$$(h^2 P - z) Op_h(q_\kappa(z, h)) = I + h^N Op_h(\tilde{r}_{\kappa,N}(z, h)), \quad (3.4)$$

where $q_\kappa(z, h) = \sum_{j=0}^{N-1} h^j q_{\kappa,j}(z)$ with $q_{\kappa,j}(z) \in S(-2-j)$ and $\tilde{r}_{\kappa,N}(z, h) \in S(-N)$ with seminorms growing polynomially in $\langle z \rangle / \text{dist}(z, \mathbb{R}^+)$ uniformly in $h \in (0, 1]$. On the other hand, we can write

$$(h^2 P_\kappa - z) \tilde{\chi}_\kappa \text{Op}_h(q_\kappa(z, h)) \chi_\kappa = \tilde{\chi}_\kappa (h^2 P_\kappa - z) \text{Op}_h(q_\kappa(z, h)) \chi_\kappa + [h^2 P_\kappa, \tilde{\chi}_\kappa] \text{Op}_h(q_\kappa(z, h)) \chi_\kappa. \quad (3.5)$$

Here $[h^2 P_\kappa, \tilde{\chi}_\kappa]$ and χ_κ have coefficients with disjoint supports. Thanks to (3.4) and the composition of pseudo-differential operators with disjoint supports, we have

$$(h^2 P_\kappa - z) \tilde{\chi}_\kappa \text{Op}_h(q_\kappa(z, h)) \chi_\kappa = \chi_\kappa + h^N \tilde{\chi}_\kappa \text{Op}_h(r_{\kappa,N}(z, h)) \chi_\kappa,$$

with $r_{\kappa,N}(z, h)$ satisfying the required property. We then compose to the right and the left of above equality with κ^* and κ_* respectively and get

$$(h^2 P - z) \tilde{\phi}_\kappa \text{Op}_h^\kappa(q_\kappa(z, h)) \phi_\kappa = \phi_\kappa + h^N \tilde{\phi}_\kappa \text{Op}_h^\kappa(r_{\kappa,N}(z, h)) \phi_\kappa.$$

This gives the result and the proof is complete. \square

Next, we give an application of the parametrix given in Proposition 3.1 and have the following result (see [5], [8]).

Proposition 3.2. *Let $\phi_\kappa, \tilde{\phi}_\kappa, \tilde{\tilde{\phi}}_\kappa$ be as in Proposition 3.1 and $f \in C_0^\infty(\mathbb{R})$. Then for all $N \geq 1$ and all $h \in (0, 1]$,*

$$f(h^2 P) \phi_\kappa = \sum_{j=0}^{N-1} h^j \tilde{\phi}_\kappa \text{Op}_h^\kappa(a_{\kappa,j}) \phi_\kappa + h^N R_{\kappa,N}(h), \quad (3.6)$$

where $a_{\kappa,j} \in S(-\infty)$ with $\text{supp}(a_{\kappa,j}) \subset \text{supp}(f \circ p_\kappa)$ for $j = 0, \dots, N-1$. Moreover, for all $m \geq 0$, there exists $C > 0$ such that for all $h \in (0, 1]$,

$$\|R_N(h)\|_{\mathcal{L}(H^{-m}(M), H^m(M))} \leq Ch^{-2m}. \quad (3.7)$$

Proof. The proof is essentially given in [8]. For the reader's convenience, we recall the main steps. By using Proposition 3.1 and the Helffer-Sjöstrand formula (see [15]), namely

$$f(h^2 P) = -\frac{1}{\pi} \int_{\mathbb{C}} \bar{\partial} \tilde{f}(z) (h^2 P - z)^{-1} dL(z),$$

where \tilde{f} is an almost analytic extension of f , the Cauchy formula implies (3.6) with

$$R_{\kappa,N}(h) = \frac{1}{\pi} \int_{\mathbb{C}} \bar{\partial} \tilde{f}(z) (h^2 P - z)^{-1} \tilde{\phi}_\kappa \text{Op}_h^\kappa(r_{\kappa,N}(z, h)) \phi_\kappa dL(z).$$

It remains to prove (3.7). This leads to study the action on $L^2(\mathbb{R}^d)$ of the map

$$\int_{\mathbb{C}} \bar{\partial} \tilde{f}(z) (P_\kappa + 1)^{m/2} (h^2 P_\kappa - z)^{-1} \tilde{\chi}_\kappa \text{Op}_h(r_{\kappa,N}(z, h)) \chi_\kappa (P_\kappa + 1)^{m/2} dL(z).$$

Using a trick as in (3.5), we can find a globally defined operator P which coincides with P_κ on the support of $\tilde{\chi}_\kappa$. We see that $\|(h^2 P - z)^{-1}\|_{\mathcal{L}(L^2(\mathbb{R}^d))} \leq C |\text{Im } z|^{-1}$ and

$$(P + 1)^{m/2} \text{Op}_h(r_{\kappa,N}(z, h)) \chi_\kappa (P + 1)^{m/2} = h^{-2m} \text{Op}_h(\tilde{r}_{\kappa,N}(z, h)),$$

where $\tilde{r}_{\kappa,N}(z, h) \in S(-N+2m)$ with seminorms growing polynomially in $1/\text{dist}(z, \mathbb{R}^+)$ uniformly in $h \in (0, 1]$ which are harmless since \tilde{f} is compactly supported and $\bar{\partial} \tilde{f}(z) = O(|\text{Im } z|^\infty)$. By choosing N such that $N - 2m > d$, the result then follows from the $\mathcal{L}(L^2(\mathbb{R}^d))$ bound of pseudo-differential operator given in Proposition 2.1. \square

A direct consequence of Proposition 2.2 using partition of unity and Proposition 2.1 is the following result. (see [8], Corollary 2.2 or [5]).

Corollary 3.3. *Let $f \in C_0^\infty(\mathbb{R})$. Then for all $1 \leq q \leq r \leq \infty$, there exists $C > 0$ such that for all $h \in (0, 1]$,*

$$\|f(h^2 P)\|_{\mathcal{L}(L^q(M), L^r(M))} \leq Ch^{-(\frac{d}{q} - \frac{d}{r})}.$$

The next proposition gives the Littlewood-Paley decomposition on compact manifolds without boundary (see [8], Corollary 2.3) which is similar to Proposition 2.4.

Proposition 3.4. *There exist $\varphi_0 \in C_0^\infty(\mathbb{R})$ and $\varphi \in C_0^\infty(\mathbb{R} \setminus \{0\})$ such that for all $q \in [2, \infty)$, there exists $C > 0$,*

$$\|u\|_{L^q(M)} \leq C \left(\sum_{h^{-1} \cdot \text{dya}} \|\varphi(h^2 P)u\|_{L^q(M)}^2 \right)^{1/2} + C\|u\|_{L^2(M)},$$

for all $u \in C_0^\infty(M)$.

3.3 Reduction of problem

In this subsection, we firstly show how to get Corollary 1.4 from Theorem 1.2 and then give a reduction of Theorem 1.2.

Proof of Corollary 1.4. Since we are working on compact manifolds without boundary, it is well-known that there exists an orthonormal basis $(e_j)_{j \in \mathbb{N}}$ of $L^2(M) := L^2(M, d\text{vol}_g)$ of C^∞ functions on M such that

$$|D_g|^\sigma e_j = \lambda_j^\sigma e_j,$$

with $0 \leq \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \dots$, $\lim_{j \rightarrow \infty} \lambda_j = +\infty$. For any f a piecewise continuous function, the functional $f(|D_g|)$ is defined as

$$f(|D_g|)u := \sum_{j \in \mathbb{N}} f(\lambda_j) u_j e_j.$$

If we set $j_0 := \dim(\ker |D_g|^\sigma)$, then $\lambda_0 = \lambda_1 = \dots = \lambda_{j_0-1} = 0$ and $\lambda_j \geq \lambda_{j_0} > 0$ for $j \geq j_0$. Here the number j_0 stands for the number of connected components of M and the corresponding eigenfunctions $(e_j)_{j=0}^{j_0-1}$ are constant functions. We now define the projection on $\ker(|D_g|^\sigma)$ by

$$\Pi_0 u := \sum_{j < j_0} u_j e_j, \quad \text{where } u_j := \langle e_j, u \rangle_{L^2(M)} = \int_M \overline{e_j(m)} u(m) d\text{vol}_g(m).$$

By the Duhamel formula, the equation (1.10) can be written as

$$v(t) = \cos(t|D_g|^\sigma) v_0 + \frac{\sin(t|D_g|^\sigma)}{|D_g|^\sigma} v_1 + \int_0^t \frac{\sin((t-s)|D_g|^\sigma)}{|D_g|^\sigma} G(s) ds.$$

We remark that the only problem may happen on $\ker(|D_g|^\sigma)$ of $\frac{\sin(t|D_g|^\sigma)}{|D_g|^\sigma}$. But it is not the case because

$$\Pi_0 \frac{\sin(t|D_g|^\sigma)}{|D_g|^\sigma} v_1 = \sum_{j < j_0} \frac{\sin(t\lambda_j^\sigma)}{\lambda_j^\sigma} v_{1,j} e_j = \sum_{j < j_0} t \frac{\sin(t\lambda_j^\sigma)}{t\lambda_j^\sigma} v_{1,j} e_j = t \sum_{j < j_0} v_{1,j} e_j = t \Pi_0 v_1.$$

Since $\ker(|D_g|^\sigma)$ is generated by constant functions, the local in time Strichartz estimates of $\Pi_0 v$, namely $\|\Pi_0 v\|_{L^p(I, L^q(M))}$ with I a bounded interval, can be controlled by any Sobolev norms of data. Therefore, we only need to study the local in time Strichartz of v away from $\ker(|D_g|^\sigma)$. Using the fact that

$$\cos(t|D_g|^\sigma) = \frac{e^{it|D_g|^\sigma} + e^{-it|D_g|^\sigma}}{2}, \quad \sin(t|D_g|^\sigma) = \frac{e^{it|D_g|^\sigma} - e^{-it|D_g|^\sigma}}{2i},$$

the Strichartz estimates (1.11) follow directly from the ones of $e^{\pm it|D_g|^\sigma}$ as in (1.8). This gives Corollary 1.4. \square

We now prove Theorem 1.2. To do so, we have the following reduction.

Proposition 3.5. *Consider (M, g) a smooth compact Riemannian manifold of dimension $d \geq 1$. Let $\sigma \in (0, \infty) \setminus \{1\}$ and $\varphi \in C_0^\infty(\mathbb{R} \setminus \{0\})$. If there exists $t_0 > 0$ small enough and $C > 0$ such that for all $u_0 \in L^1(M)$ and all $h \in (0, 1]$,*

$$\|e^{-ith^{-1}(h|D_g|)^\sigma} \varphi(h^2 P) u_0\|_{L^\infty(M)} \leq C h^{-d/2} |t|^{-d/2} \|u_0\|_{L^1(M)}, \quad (3.8)$$

for all $t \in [-t_0, t_0]$, then Theorem 1.2 holds true.

Proof. The proof of homogeneous Strichartz estimates follows similarly to the one given in Proposition 2.5. We only give the proof of (1.8), i.e. $\sigma \in (1, \infty)$, the one for $\sigma \in (0, 1)$ is completely similar. The homogeneous part follows from (1.7). It remains to prove

$$\left\| \int_0^t e^{-i(t-s)|D_g|^\sigma} F(s) ds \right\|_{L^p(I, L^q(M))} \leq C \|F\|_{L^1(I, H^{\gamma_{pq} + (\sigma-1)/p}(M))}. \quad (3.9)$$

The estimate (3.9) follows easily from (1.7) and the Minkowski inequality (see [8], Corollary 2.10). Indeed, the left hand side reads

$$\begin{aligned} \left\| \int_I \mathbb{1}_{[0,t]}(s) e^{-i(t-s)|D_g|^\sigma} F(s) ds \right\|_{L^p(I, L^q(M))} &\leq \int_I \|\mathbb{1}_{[0,t]}(s) e^{-i(t-s)|D_g|^\sigma} F(s)\|_{L^p(I, L^q(M))} ds \\ &\leq \int_I \|e^{-i(t-s)|D_g|^\sigma} F(s)\|_{L^p(I, L^q(M))} ds \\ &\leq C \int_I \|F(s)\|_{H^{\gamma_{pq} + (\sigma-1)/p}(M)} ds. \end{aligned}$$

This gives (3.9) and the proof of Proposition 3.5 is complete. \square

3.4 Dispersive estimates

This subsection devotes to prove the dispersive estimates (3.8). Again thanks to the localization φ , we can replace $(h|D_g|)^\sigma$ by $\psi(h^2 P)$ where $\psi(\lambda) = \tilde{\varphi}(\lambda) \sqrt{\lambda}^\sigma$ with $\tilde{\varphi} \in C_0^\infty(\mathbb{R} \setminus \{0\})$ such that $\tilde{\varphi} = 1$ near $\text{supp}(\varphi)$. The partition of unity allows us to consider only on a local coordinates, namely $\sum_\kappa e^{-ith^{-1}\psi(h^2 P)} \varphi(h^2 P) \phi_\kappa$. By using the same argument as in Remark 2.8 and Proposition 3.2, the study of $e^{-ith^{-1}\psi(h^2 P)} \varphi(h^2 P) \phi_\kappa$ is reduced to the one of $e^{-ith^{-1}\psi(h^2 P)} \tilde{\phi}_\kappa \text{Op}_h^\kappa(a_\kappa) \phi_\kappa$ with $a_\kappa \in S(-\infty)$ and $\text{supp}(a_\kappa) \subset \text{supp}(\varphi \circ p_\kappa)$. Let us set

$$u(t) = e^{-ith^{-1}\psi(h^2 P)} \tilde{\phi}_\kappa \text{Op}_h^\kappa(a_\kappa) \phi_\kappa u_0.$$

We see that u solves the following semi-classical evolution equation

$$\begin{cases} (hD_t + \psi(h^2 P))u(t) &= 0, \\ u|_{t=0} &= \tilde{\phi}_\kappa \text{Op}_h^\kappa(a_\kappa) \phi_\kappa u_0. \end{cases} \quad (3.10)$$

The WKB method allows us to construct an approximation of the solution to (3.10) in finite time independent of h . To do so, we firstly choose $\phi'_\kappa, \tilde{\phi}'_\kappa, \tilde{\tilde{\phi}}'_\kappa \in C_0^\infty(U_\kappa)$ such that $\phi'_\kappa = 1$ near $\text{supp}(\tilde{\phi}_\kappa)$ (see Proposition 3.1 for $\tilde{\phi}_\kappa$), $\tilde{\phi}'_\kappa = 1$ near $\text{supp}(\phi'_\kappa)$ and $\tilde{\tilde{\phi}}'_\kappa = 1$ near $\text{supp}(\tilde{\phi}'_\kappa)$. Proposition 3.2 then implies

$$\psi(h^2 P) \phi'_\kappa = \tilde{\phi}'_\kappa \text{Op}_h^\kappa(b_\kappa(h)) \phi'_\kappa + h^N R'_{\kappa, N}(h), \quad (3.11)$$

where $b_\kappa(h) = \sum_{l=1}^{N-1} h^l b_{\kappa, l}$ with $b_{\kappa, l} \in S(-\infty)$ and $R'_{\kappa, N}(h) = O_{\mathcal{L}(L^2(M))}(1)$. By using the global extension operator defined in (3.2), we can apply the construction of the WKB approximation given in Subsection 2.2 and find $t_0 > 0$ small enough, a function $S_\kappa \in C^\infty([-t_0, t_0] \times \mathbb{R}^{2d})$ and a

sequence $a_{\kappa,j}(t, \cdot, \cdot) \in S(-\infty)$ satisfying $\text{supp}(a_{\kappa,j}(t, \cdot, \cdot)) \subset p^{-1}(J)$ (see (3.3) for the definition of p) for some small neighborhood J of $\text{supp}(\varphi)$ not containing the origin uniformly in $t \in [-t_0, t_0]$ such that

$$(hD_t + \text{Op}_h(b_\kappa(h)))J_{\kappa,N}(t) = R_{\kappa,N}(t), \quad (3.12)$$

where

$$J_{\kappa,N}(t) := \sum_{j=0}^{N-1} h^j J_h(S_\kappa(t), a_{\kappa,j}(t)), \quad J_N(0) = \text{Op}_h(a_\kappa),$$

satisfying for all $t \in [-t_0, t_0]$ and all $(x, \xi) \in p^{-1}(J)$,

$$|\partial_x^\alpha \partial_\xi^\beta (S_\kappa(t, x, \xi) - x \cdot \xi)| \leq C_{\alpha\beta} |t|, \quad |\alpha + \beta| \geq 1, \quad (3.13)$$

$$\left| \partial_x^\alpha \partial_\xi^\beta (S_\kappa(t, x, \xi) - x \cdot \xi + t \sqrt{p(x, \xi)}^\sigma) \right| \leq C_{\alpha\beta} |t|^2, \quad (3.14)$$

and for all $h \in (0, 1]$,

$$\|J_{\kappa,N}(t)\|_{\mathcal{L}(L^1(\mathbb{R}^d), L^\infty(\mathbb{R}^d))} \leq Ch^{-d/2} |t|^{-d/2}, \quad (3.15)$$

$$R_{\kappa,N}(t) = O_{\mathcal{L}(L^2(\mathbb{R}^d))}(h^{N+1}). \quad (3.16)$$

Next, we need the following micro-local finite propagation speed.

Lemma 3.6. *Let $\sigma \in (0, \infty) \setminus \{1\}$, $\chi, \tilde{\chi} \in C_0^\infty(\mathbb{R}^d)$ such that $\tilde{\chi} = 1$ near $\text{supp}(\chi)$, $a(t) \in S(-\infty)$ with $\text{supp}(a(t, \cdot, \cdot)) \subset p^{-1}(J)$ uniformly in $t \in [-t_0, t_0]$ and $S \in C^\infty([-t_0, t_0] \times \mathbb{R}^{2d})$ satisfy (3.14) for all $t \in [-t_0, t_0]$ and all $(x, \xi) \in p^{-1}(J)$. Then for $t_0 > 0$ small enough,*

$$J_h(S(t), a(t))\chi = \tilde{\chi} J_h(S(t), a(t))\chi + \tilde{R}(t),$$

where $\tilde{R}(t) = O_{\mathcal{L}(L^2(\mathbb{R}^d))}(h^\infty)$.

Proof. The kernel of $J_h(S(t), a(t))\chi - \tilde{\chi} J_h(S(t), a(t))\chi$ is given by

$$K_h(t, x, y) = (2\pi h)^{-d} \int_{\mathbb{R}^d} e^{ih^{-1}(S(t, x, \xi) - y\xi)} (1 - \tilde{\chi})(x) a(t, x, \xi) \chi(y) d\xi.$$

Using (3.14), we can write for $t_0 > 0$ small enough, $t \in [-t_0, t_0]$ and $(x, \xi) \in p^{-1}(J)$,

$$S(t, x, \xi) - y\xi = (x - y)\xi - t \sqrt{p(x, \xi)}^\sigma + O(t^2).$$

By change of variables $\eta = \sqrt{G(x)}\xi$ or $\xi = \sqrt{g(x)}\eta$, we have

$$K_h(t, x, y) = (2\pi h)^{-d} \int_{\mathbb{R}^d} e^{ih^{-1}\Phi(t, x, y, \eta)} (1 - \tilde{\chi})(x) a(t, x, \sqrt{g(x)}\eta) \chi(y) \sqrt{\det g(x)} dx,$$

where $\Phi(t, x, y, \eta) = \sqrt{g(x)}(x - y)\eta - t|\eta|^\sigma + O(t^2)$. Thanks to the support of χ and $\tilde{\chi}$, we see that $|x - y| \geq C$. This gives for $t_0 > 0$ small enough that

$$|\nabla_\eta \Phi(t, x, y, \eta)| = |\sqrt{g(x)}(x - y) - t\sigma\eta|\eta|^{\sigma-2} + O(t^2)| \geq C(1 + |x - y|).$$

Here we also use the fact that $||\sqrt{g(x)}||$ is bounded from below and above (see (3.3)). Using the fact that for all $\beta \in \mathbb{N}^d$ satisfying $|\beta| \geq 2$,

$$|\partial_\eta^\beta \Phi(t, x, y, \eta)| \leq C_\beta,$$

the non stationary phase theorem implies for all $N \geq 1$, all $t \in [-t_0, t_0]$ and all $x, y \in \mathbb{R}^d$,

$$|K_h(t, x, y)| \leq Ch^{N-d} (1 + |x - y|)^{-N}.$$

The Schur's Lemma gives $\tilde{R}(t) = O_{\mathcal{L}(L^2(\mathbb{R}^d))}(h^\infty)$. This ends the proof. \square

Proof of dispersive estimates (3.8). With the same spirit as in (3.1), let us set $J_N^\kappa(t) = \kappa^* J_{\kappa,N}(t) \kappa_*$, $R_N^\kappa(t) = \kappa^* R_{\kappa,N}(t) \kappa_*$ where $J_{\kappa,N}(t)$ and $R_{\kappa,N}(t)$ given in (3.12). The Duhamel formula gives

$$\begin{aligned} u(t) &= e^{-ith^{-1}\psi(h^2P)} \tilde{\phi}_\kappa Op_h^\kappa(a_\kappa) \phi_\kappa u_0 \\ &= \tilde{\phi}_\kappa J_N^\kappa(t) \phi_\kappa u_0 - ih^{-1} \int_0^t e^{-i(t-s)h^{-1}\psi(h^2P)} (hD_s + \psi(h^2P)) \tilde{\phi}_\kappa J_N^\kappa(s) \phi_\kappa u_0 ds. \end{aligned}$$

We also have from (3.11) that

$$(hD_s + \psi(h^2P)) \tilde{\phi}_\kappa J_N^\kappa(s) \phi_\kappa = \tilde{\phi}_\kappa hD_s J_N^\kappa(s) \phi_\kappa + \tilde{\phi}_\kappa' Op_h^\kappa(b_\kappa(h)) \tilde{\phi}_\kappa J_N^\kappa(s) \phi_\kappa + h^N R'_{\kappa,N}(h) \tilde{\phi}_\kappa J_N^\kappa(s) \phi_\kappa.$$

The micro-local finite propagation speed given in Lemma 3.6 and (3.12) imply

$$\begin{aligned} (hD_s + \psi(h^2P)) \tilde{\phi}_\kappa J_N^\kappa(s) \phi_\kappa &= \tilde{\phi}_\kappa' \kappa^* (hD_s + Op_h(b_\kappa(h))) J_N(s) \kappa_* \phi_\kappa + \tilde{R}_\kappa(s) + h^N R'_{\kappa,N}(h) \tilde{\phi}_\kappa J_N^\kappa(s) \phi_\kappa \\ &= \tilde{\phi}_\kappa' R_N^\kappa(s) \phi_\kappa + \tilde{R}_\kappa(s) + h^N R'_{\kappa,N}(h) \tilde{\phi}_\kappa J_N^\kappa(s) \phi_\kappa, \end{aligned}$$

where $\tilde{R}_\kappa(s) = O_{\mathcal{L}(L^2(M))}(h^\infty)$. Here we also use the L^2 -boundedness of pseudo-differential operators with symbols in $S(-\infty)$. We then get

$$u(t) = \tilde{\phi}_\kappa J_N^\kappa(t) \phi_\kappa u_0 + \mathcal{R}_N^\kappa(t) u_0,$$

where

$$\mathcal{R}_N^\kappa(t) u_0 = -ih^{-1} \int_0^t e^{-i(t-s)h^{-1}\psi(h^2P)} (\tilde{\phi}_\kappa' R_N^\kappa(s) \phi_\kappa + \tilde{R}_\kappa(s) + h^N R'_{\kappa,N}(h) \tilde{\phi}_\kappa J_N^\kappa(s) \phi_\kappa) u_0 ds.$$

By the same process as in Remark 2.8 using (3.15) and the fact that $\mathcal{R}_N^\kappa(t) = O_{\mathcal{L}(L^2(M))}(h^{N-1})$ for all $t \in [-t_0, t_0]$, we obtain

$$\|e^{-ith^{-1}\psi(h^2P)} \varphi(h^2P) \phi_\kappa u_0\|_{L^\infty(M)} \leq Ch^{-d/2} |t|^{-d/2} \|u_0\|_{L^1(M)},$$

for all $t \in [-t_0, t_0]$. The dispersive estimates (3.8) then follow from the above estimates and partition of unity. This completes the proof. \square

4 Nonlinear applications

In this section, we give the proofs of Theorem 1.5 and Corollary 1.6 and Theorem 1.7.

Proof of Theorem 1.5. We only treat the case $\sigma \in (1, \infty)$ where we have Strichartz estimates with loss of derivatives. The one for $\sigma \in (0, 1)$ is similar and essentially given in [14], Theorem 1.7. We follow the standard process (see e.g [16] or [8]) by using the fixed point argument in a suitable Banach space. We firstly choose $p > \max(\nu - 1, 4)$ when $d = 1$ and $p > \max(\nu - 1, 2)$ when $d \geq 2$ such that $\gamma > d/2 - 1/p$ and then choose $q \geq 2$ such that

$$\frac{2}{p} + \frac{d}{q} = \frac{d}{2}.$$

Let us consider

$$X_T := \left\{ u \in C(I, H^\gamma(M)) \cap L^p(I, H_q^\alpha(M)), \|u\|_{L^\infty(I, H^\gamma(M))} + \|u\|_{L^p(I, H_q^\gamma(M))} \leq N \right\}$$

equipped with the distance

$$\|u - v\|_{X_T} := \|u - v\|_{L^\infty(I, L^2(M))} + \|u - v\|_{L^p(I, H_q^{-\gamma pq - (\sigma-1)/p}(M))},$$

where $I = [0, T]$, $T, N > 0$ will be chosen later and $\alpha = \gamma - \gamma pq - (\sigma - 1)/p$. Here $H_q^\gamma(M) := (1 - \Delta_g)^{-\gamma/2} L^q(M)$ is the generalized Sobolev space on M and $H^\gamma(M) := H_2^\gamma(M)$. Using the

persistence of regularity (see [9], Theorem 1.25), we have $(X_T, \|\cdot\|_{X_T})$ is a complete metric space. By the Duhamel formula, it suffices to prove that the functional

$$\Phi_{u_0}(u)(t) = e^{-it|D_g|^\sigma} u_0 + i\mu \int_0^t e^{-i(t-s)|D_g|^\sigma} |u(s)|^{\nu-1} u(s) ds$$

is a contraction on X_T . The Strichartz estimates (1.8) imply

$$\|\Phi_{u_0}(u)\|_{L^\infty(I, H^\gamma(M))} + \|\Phi_{u_0}(u)\|_{L^p(I, H_q^\alpha(M))} \lesssim \|u_0\|_{H^\gamma(M)} + \|F(u)\|_{L^1(I, H^\gamma(M))},$$

where $F(u) = -\mu|u|^{\nu-1}u$. Using our assumption on ν (i.e. ν is an odd integer or (1.13) otherwise), the fractional derivatives (see e.g. [22], Appendix) and Hölder inequality, we have

$$\begin{aligned} \|F(u)\|_{L^1(I, H^\gamma(M))} &\lesssim \left\| |u(\cdot)|^{\nu-1} u(\cdot) \right\|_{L^1(I)} \lesssim \|u\|_{L^{\nu-1}(I, L^\infty(M))}^{\nu-1} \|u\|_{L^\infty(I, H^\gamma(M))} \\ &\lesssim T^{1-\frac{\nu-1}{p}} \|u\|_{L^p(I, L^\infty(M))}^{\nu-1} \|u\|_{L^\infty(I, H^\gamma(M))}. \end{aligned}$$

Note that by working in local coordinates, the fractional derivatives on compact manifold are reduced to the ones on \mathbb{R}^d . Similarly, using the fact that for all $z, \zeta \in \mathbb{C}$,

$$|F(z) - F(\zeta)| \lesssim |z - \zeta|(|z|^{\nu-1} + |\zeta|^{\nu-1}), \quad (4.1)$$

we have

$$\begin{aligned} \|F(u) - F(v)\|_{L^1(I, L^2(M))} &\lesssim \left(\|u\|_{L^{\nu-1}(I, L^\infty(M))}^{\nu-1} + \|v\|_{L^{\nu-1}(I, L^\infty(M))}^{\nu-1} \right) \|u - v\|_{L^\infty(I, L^2(M))} \\ &\lesssim T^{1-\frac{\nu-1}{p}} \left(\|u\|_{L^p(I, L^\infty(M))}^{\nu-1} + \|v\|_{L^p(I, L^\infty(M))}^{\nu-1} \right) \|u - v\|_{L^\infty(I, L^2(M))}. \end{aligned}$$

The Sobolev embedding with $\alpha > d/q$ implies $L^p(I, H_q^\alpha(M)) \subset L^p(I, L^\infty(M))$. Thus,

$$\begin{aligned} \|\Phi_{u_0}(u)\|_{L^\infty(I, H^\gamma(M))} + \|\Phi_{u_0}(u)\|_{L^p(I, H_q^\alpha(M))} \\ \lesssim \|u_0\|_{H^\gamma(M)} + T^{1-\frac{\nu-1}{p}} \|u\|_{L^p(I, H_q^\alpha(M))}^{\nu-1} \|u\|_{L^\infty(I, H^\gamma(M))}, \end{aligned}$$

and

$$\begin{aligned} \|\Phi_{u_0}(u) - \Phi_{u_0}(v)\|_{L^\infty(I, L^2(M))} + \|\Phi_{u_0}(u) - \Phi_{u_0}(v)\|_{L^p(I, H_q^{-\gamma p q - (\sigma-1)/p}(M))} \\ \lesssim T^{1-\frac{\nu-1}{p}} \left(\|u\|_{L^p(I, L^\infty(M))}^{\nu-1} + \|v\|_{L^p(I, L^\infty(M))}^{\nu-1} \right) \|u - v\|_{L^\infty(I, L^2(M))} \quad (4.2) \\ \lesssim T^{1-\frac{\nu-1}{p}} \left(\|u\|_{L^p(I, H_q^\alpha(M))}^{\nu-1} + \|v\|_{L^p(I, H_q^\alpha(M))}^{\nu-1} \right) \|u - v\|_{L^\infty(I, L^2(M))}. \end{aligned}$$

This implies for all $u, v \in X_T$, there exists $C > 0$ independent of $u_0 \in H^\gamma(M)$ such that

$$\|\Phi_{u_0}(u)\|_{L^\infty(I, H^\gamma(M))} + \|\Phi_{u_0}(u)\|_{L^p(I, H_q^\alpha(M))} \leq C\|u_0\|_{H^\gamma(M)} + CT^{1-\frac{\nu-1}{p}} N^\nu,$$

and

$$\|\Phi_{u_0}(u) - \Phi_{u_0}(v)\|_{X_T} \leq CT^{1-\frac{\nu-1}{p}} N^{\nu-1} \|u - v\|_{X_T}.$$

Therefore, if we set $N = 2C\|u_0\|_{H^\gamma(M)}$ and choose $T > 0$ small enough so that $CT^{1-\frac{\nu-1}{p}} N^{\nu-1} \leq \frac{1}{2}$, then X_T is stable by Φ_{u_0} and Φ_{u_0} is a contraction on X_T . The fixed point theorem gives the existence of solution $u \in C(I, H^\gamma(M)) \cap L^p(I, L^\infty(M))$ to (NLFS). It remains to show the uniqueness. Consider $u, v \in C(I, H^\gamma(M)) \cap L^p(I, L^\infty(M))$ two solutions of (NLFS). Since the uniqueness is a local property (see also [9]), it suffices to show $u = v$ for T is small. Using (4.2), we have

$$\|u - v\|_{X_T} \leq CT^{1-\frac{\nu-1}{p}} \left(\|u\|_{L^p(I, L^\infty(M))}^{\nu-1} + \|v\|_{L^p(I, L^\infty(M))}^{\nu-1} \right) \|u - v\|_{X_T}.$$

Since $\|u\|_{L^p(I, L^\infty(M))}$ is small if T is small and similarly for v , we see that if $T > 0$ small enough,

$$\|u - v\|_{X_T} \leq \frac{1}{2} \|u - v\|_{X_T}.$$

This completes the proof. \square

Proof of Corollary 1.6. By the assumptions given in Corollary 1.6, we apply Theorem 1.5 with $\gamma = \sigma/2$ and see that for all $u_0 \in H^{\sigma/2}(M)$, there exist $T > 0$ and a unique solution $u \in C([0, T], H^{\sigma/2}(M)) \cap L^p([0, T], L^\infty(M))$ to the defocusing (NLFS). Note that the time T depends only on $\|u_0\|_{H^{\sigma/2}(M)}$. Moreover, by a classical approximation argument, the following quantities are conserved for $u_0 \in H^{\sigma/2}(M)$,

$$\begin{aligned} \|u(t)\|_{L^2(M)}^2 &= M(u_0), \\ \frac{1}{2} \| |D_g|^{\sigma/2} u(t) \|_{L^2(M)}^2 + \frac{1}{\nu+1} \|u(t)\|_{L^{\nu+1}(M)}^{\nu+1} &= E(u_0). \end{aligned}$$

This shows that $\|u(t)\|_{H^{\sigma/2}(M)}$ remains bounded for all t in the existence domain. Thus we can apply Theorem 1.5 again with the initial data starting at T and obtain a unique solution $u \in C([0, 2T], H^{\sigma/2}(M)) \cap L^p([0, 2T], L^\infty(M))$. By repeating this process, we extend the solution for positive times. Similarly, the same result holds for negative times. This ends the proof. \square

Proof of Theorem 1.7. The proof is very close to the one of Theorem 1.5. We only consider the case $\sigma \in (1, \infty)$, the one for $\sigma \in (0, 1)$ is similar (see also [14], Theorem 1.13). Let (p, q) and α be as in the proof of Theorem 1.5. We will solve (NLFW) in

$$\begin{aligned} Y_T := \left\{ v \in C(I, H^\gamma(M)) \cap C^1(I, H^{\gamma-\sigma}(M)) \cap L^p(I, H_q^\alpha(M)), \right. \\ \left. \| [v] \|_{L^\infty(I, H^\gamma(M))} + \| v \|_{L^p(I, H_q^\alpha(M))} \leq N \right\} \end{aligned}$$

equipped with the distance

$$\|v - w\|_{Y_T} := \| [v - w] \|_{L^\infty(I, L^2(M))} + \| v - w \|_{L^p(I, H_q^{-\gamma p q - (\sigma-1)/p}(M))},$$

where $I = [0, T]$ and $T, N > 0$ will be chosen later. Here we denote

$$\| [v] \|_{L^\infty(I, H^\gamma(M))} = \| v \|_{L^\infty(I, H^\gamma(M))} + \| \partial_t v \|_{L^\infty(I, H^{\gamma-\sigma}(M))}.$$

The persistence of regularity implies that $(Y_T, \| \cdot \|_{Y_T})$ is a complete metric space. By the Duhamel formula, it suffices to prove that the functional

$$\Phi_{v_0, v_1}(v)(t) = \cos(t|D|^\sigma) v_0 + \frac{\sin(t|D|^\sigma)}{|D|^\sigma} v_1 - \mu \int_0^t \frac{\sin((t-s)|D|^\sigma)}{|D|^\sigma} |v(s)|^{\nu-1} v(s) ds \quad (4.3)$$

is a contraction on Y_T . The local Strichartz estimates (1.11) imply

$$\begin{aligned} \| [\Phi_{v_0, v_1}(v)] \|_{L^\infty(I, H^\gamma(M))} + \| \Phi_{v_0, v_1}(v) \|_{L^p(I, H_q^\alpha(M))} &\lesssim \| [v](0) \|_{H^\gamma(M)} + \| F(v) \|_{L^1(I, H^{\gamma-\sigma}(M))} \\ &\lesssim \| [v](0) \|_{H^\gamma(M)} + \| F(v) \|_{L^1(I, H^\gamma(M))}. \end{aligned}$$

As in the proof of Theorem 1.5, the fractional derivatives with the assumption on ν given in Theorem 1.7, the Hölder inequality imply

$$\| F(v) \|_{L^1(I, H^\gamma(M))} \lesssim T^{1-\frac{\nu-1}{p}} \| v \|_{L^p(I, L^\infty(M))}^{\nu-1} \| v \|_{L^\infty(I, H^\gamma(M))}.$$

Similarly, using (4.1), we have

$$\| F(v) - F(w) \|_{L^1(I, L^2(M))} \lesssim T^{1-\frac{\nu-1}{p}} \left(\| v \|_{L^p(I, L^\infty(M))}^{\nu-1} + \| w \|_{L^p(I, L^\infty(M))}^{\nu-1} \right) \| u - v \|_{L^\infty(I, L^2(M))}.$$

The Sobolev embedding $L^p(I, H_q^\alpha(M)) \subset L^p(I, L^\infty(M))$ then implies that

$$\begin{aligned} \| [\Phi_{v_0, v_1}(v)] \|_{L^\infty(I, H^\gamma(M))} + \| \Phi_{v_0, v_1}(v) \|_{L^p(I, H_q^\alpha(M))} \\ \lesssim \| [v](0) \|_{H^\gamma(M)} + T^{1-\frac{\nu-1}{p}} \| v \|_{L^p(I, H_q^\alpha(M))}^{\nu-1} \| v \|_{L^\infty(I, H^\gamma(M))}, \end{aligned}$$

and

$$\begin{aligned} \|\Phi_{v_0, v_1}(v) - \Phi_{v_0, v_1}(w)\|_{Y_T} &\lesssim T^{1-\frac{\nu-1}{p}} \left(\|v\|_{L^p(I, L^\infty(M))}^{\nu-1} + \|w\|_{L^p(I, L^\infty(M))}^{\nu-1} \right) \|u - v\|_{Y_T} \\ &\lesssim T^{1-\frac{\nu-1}{p}} \left(\|v\|_{L^p(I, H_q^\alpha(M))}^{\nu-1} + \|w\|_{L^p(I, H_q^\alpha(M))}^{\nu-1} \right) \|u - v\|_{Y_T}. \end{aligned} \quad (4.4)$$

Therefore, for all $v, w \in Y_T$, there exists a constant $C > 0$ independent of v_0, v_1 such that

$$\|[\Phi_{v_0, v_1}(v)]\|_{L^\infty(I, H^\gamma(M))} + \|\Phi_{v_0, v_1}(v)\|_{L^p(I, H_q^\alpha(M))} \leq C\|[v](0)\|_{H^\gamma(M)} + CT^{1-\frac{\nu-1}{p}}N^\nu,$$

and

$$\|\Phi_{v_0, v_1}(v) - \Phi_{v_0, v_1}(w)\|_{Y_T} \leq CT^{1-\frac{\nu-1}{p}}N^{\nu-1}\|u - v\|_{Y_T}.$$

Setting $N = 2C\|[v](0)\|_{H^\gamma(M)}$ and choosing $T > 0$ small enough so that $CT^{1-\frac{\nu-1}{p}}N^{\nu-1} \leq \frac{1}{2}$, we see that Y_T is stable by Φ_{v_0, v_1} and Φ_{v_0, v_1} is a contraction on Y_T . By the fixed point theorem, there exists a unique solution $v \in Y_T$ to (NLFW). The uniqueness of solution $v \in C(I, H^\gamma(M)) \cap C^1(I, H^{\gamma-\sigma}(M)) \cap L^p(I, L^\infty(M))$ follows as in the proof of Theorem 1.5 using (4.4). \square

A Hamilton-Jacobi equation

In this appendix, we will recall the standard Hamilton-Jacobi equation (see e.g. [27]). Let us consider the following Hamilton-Jacobi equation

$$\begin{cases} \partial_t S(t, x, \xi) + H(x, \nabla_x S(t, x, \xi)) &= 0, \\ S(0, x, \xi) &= x \cdot \xi, \end{cases} \quad (A.1)$$

where $H \in C^\infty(\mathbb{R}^{2d})$ satisfies that for all $\alpha, \beta \in \mathbb{N}^d$, there exists $C_{\alpha\beta} > 0$ such that for all $x, \xi \in \mathbb{R}^d$,

$$|\partial_x^\alpha \partial_\xi^\beta H(x, \xi)| \leq C_{\alpha\beta}. \quad (A.2)$$

The Hamiltonian flow associated to H is denoted by $\Phi_H(t, x, \xi) := (X(t, x, \xi), \Xi(t, x, \xi))$ where

$$\begin{cases} \dot{X}(t) &= \nabla_\xi H(X(t), \Xi(t)), \\ \dot{\Xi}(t) &= -\nabla_x H(X(t), \Xi(t)), \end{cases} \quad \text{and} \quad \begin{cases} X(0) &= x, \\ \Xi(0) &= \xi. \end{cases}$$

We have the following result (see [27]).

Lemma A.1. *Let $t_0 \geq 0$ and $\alpha, \beta \in \mathbb{N}^d$ be such that $|\alpha| + |\beta| \geq 1$. Then there exists $C_{\alpha\beta t_0} > 0$ such that for all $t \in [-t_0, t_0]$ and all $(x, \xi) \in \mathbb{R}^{2d}$,*

$$|\partial_x^\alpha \partial_\xi^\beta (\Phi_H(t, x, \xi) - (x, \xi))| \leq C_{\alpha\beta t_0} |t|.$$

Proof. The proof is essentially given in [27]. We assume first $|\alpha + \beta| = 1$ and denote

$$Z(t) = \begin{pmatrix} \partial_x X(t) & \partial_\xi X(t) \\ \partial_x \Xi(t) & \partial_\xi \Xi(t) \end{pmatrix}.$$

By direct computation, we have

$$\frac{d}{dt} Z(t) = A(t) Z(t), \quad (A.3)$$

where

$$A(t) = \begin{pmatrix} \partial_x \partial_\xi H(X(t), \Xi(t)) & \partial_\xi^2 H(X(t), \Xi(t)) \\ -\partial_x^2 H(X(t), \Xi(t)) & -\partial_\xi \partial_x H(X(t), \Xi(t)) \end{pmatrix}.$$

This implies that

$$\|Z(t) - I_{\mathbb{R}^{2d}}\| \leq \int_0^t \|A(s)\| \|Z(s)\| ds \leq M|t| + \int_0^t M \|Z(s) - I_{\mathbb{R}^{2d}}\| ds,$$

where $M = \sup_{(t,x,\xi) \in [-t_0, t_0] \times \mathbb{R}^{2d}} \|A(t)\|$. Using Gronwall inequality, we have

$$\|Z(t) - I_{\mathbb{R}^{2d}}\| \leq M|t|e^{Mt} \leq Me^{Mt_0}|t|.$$

For $|\alpha + \beta| \geq 2$, we take the derivative of (A.3) and apply again the Gronwall inequality. \square

Lemma A.2. *There exists $t_0 > 0$ small enough such that for all $t \in [-t_0, t_0]$ and all $\xi \in \mathbb{R}^d$, the map $x \mapsto X(t, x, \xi)$ is a diffeomorphism from \mathbb{R}^d onto itself. Moreover, if we denote $x \mapsto Y(t, x, \xi)$ the inverse map, then for all $t \in [-t_0, t_0]$ and all $\alpha, \beta \in \mathbb{N}^d$ satisfying $|\alpha + \beta| \geq 1$, there exists $C_{\alpha\beta} > 0$ such that for all $x, \xi \in \mathbb{R}^d$,*

$$|\partial_x^\alpha \partial_\xi^\beta (Y(t, x, \xi) - x)| \leq C_{\alpha\beta} |t|.$$

Proof. By Lemma A.1, there exists $t_0 > 0$ small enough such that

$$\|\partial_x X(t) - I_{\mathbb{R}^d}\| \leq \frac{1}{2},$$

for all $t \in [-t_0, t_0]$. By Hadamard global inversion theorem, the map $x \mapsto X(t, x, \xi)$ is a diffeomorphism from \mathbb{R}^d onto itself. Let $x \mapsto Y(t, x, \xi)$ be its inverse. By taking derivative $\partial_x^\alpha \partial_\xi^\beta$ with $|\alpha + \beta| = 1$ of the following equality

$$x = X(t, Y(t, x, \xi), \xi), \quad (\text{A.4})$$

we have

$$(\partial_x X)(t, Y(t, x, \xi), \xi) \partial_x^\alpha \partial_\xi^\beta (Y(t, x, \xi) - x) = -\partial_y^\alpha \partial_\eta^\beta (X(t, y, \eta) - y)|_{(y, \eta) = (Y(t, x, \xi), \xi)}.$$

By choosing t_0 small enough, we see that the matrix $(\partial_x X)(t, Y(t, x, \xi), \xi)$ is invertible and its inverse is bounded uniformly in $t \in [-t_0, t_0]$ and $x, \xi \in \mathbb{R}^d$. This implies that

$$|\partial_x^\alpha \partial_\xi^\beta (Y(t, x, \xi) - x)| \leq C |\partial_y^\alpha \partial_\eta^\beta (X(t, y, \eta) - y)| \leq C_{\alpha\beta} |t|.$$

For higher derivatives, we differentiate (A.4) and use an induction on $|\alpha + \beta|$. This completes the proof. \square

Now, we are able to solve the Hamilton-Jacobi equation (A.1) and have the following result.

Proposition A.3. *Let t_0 be as in Lemma A.2. Then there exists a unique function $S \in C^\infty([-t_0, t_0] \times \mathbb{R}^{2d})$ such that S solves the Hamilton-Jacobi equation (A.1). The solution S is given by*

$$S(t, x, \xi) = Y(t, x, \xi) \cdot \xi + \int_0^t (\xi \cdot \partial_\xi H - H) \circ \Phi_H(s, Y(t, x, \xi), \xi) ds, \quad (\text{A.5})$$

and S satisfies

$$\nabla_\xi S(t) = Y(t), \quad \nabla_x S(t) = \Xi(t, Y(t), \xi), \quad \Phi_H(t, \nabla_\xi S(t), \xi) = (x, \nabla_x S(t)), \quad (\text{A.6})$$

where $S(t) := S(t, x, \xi)$ and $Y(t) := Y(t, x, \xi)$. Moreover, for all $\alpha, \beta \in \mathbb{N}^d$, there exists $C_{\alpha\beta} > 0$ such that for all $t \in [-t_0, t_0]$ and all $x, \xi \in \mathbb{R}^d$,

$$|\partial_x^\alpha \partial_\xi^\beta (S(t, x, \xi) - x \cdot \xi)| \leq C_{\alpha\beta} |t|, \quad |\alpha + \beta| \geq 1, \quad (\text{A.7})$$

$$|\partial_x^\alpha \partial_\xi^\beta (S(t, x, \xi) - x \cdot \xi + tH(x, \xi))| \leq C_{\alpha\beta} |t|^2. \quad (\text{A.8})$$

Proof. It is well-known (see [27]) that the function S defined in (A.5) is the unique solution to (A.1) and satisfies (A.6). It remains to prove (A.7) and (A.8). By (A.6) and the conservation of energy, we have

$$H(x, \nabla_x S(t)) = H \circ \Phi_H(t, \nabla_\xi S(t), \xi) = H(\nabla_\xi S(t), \xi) = H(Y(t), \xi).$$

This implies that

$$S(t, x, \xi) - x \cdot \xi = t \int_0^1 \partial_t S(\theta t, x, \xi) d\theta = -t \int_0^1 H(Y(\theta t, x, \xi), \xi) d\theta.$$

Using (A.2) and Lemma A.2, we have (A.7). Next, we compute

$$\begin{aligned} \partial_t^2 S(t) &= -\partial_t [H(Y(t), \xi)] = -(\nabla_x H)(Y(t), \xi) \cdot \partial_t Y(t) \\ &= -(\nabla_x H)(Y(t), \xi) \cdot \nabla_\xi [\partial_t S(t)] = -(\nabla_x H)(Y(t), \xi) \cdot \nabla_\xi [-H(Y(t), \xi)] \\ &= (\nabla_x H)^2(Y(t), \xi) \cdot \nabla_\xi Y(t) + (\nabla_x H \cdot \nabla_\xi H)(Y(t), \xi). \end{aligned} \quad (\text{A.9})$$

The Taylor formula gives

$$\begin{aligned} S(t, x, \xi) &= x \cdot \xi - tH(x, \xi) \\ &\quad + t^2 \int_0^1 (1 - \theta) [(\nabla_x H)^2(Y(\theta t), \xi) \cdot \nabla_\xi Y(\theta t) + (\nabla_x H \cdot \nabla_\xi H)(Y(\theta t), \xi)] d\theta. \end{aligned}$$

Using again (A.2) and Lemma A.2, we have (A.8). \square

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